The Formation of Financial Networks

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Abstract

Modern banking systems are highly interconnected. Despite their various benefits, linkages between banks carry the risk of contagion. In this paper we investigate whether banks can commit ex-ante to mutually insure each other, when there is contagion risk in the financial system. Banks can share the risk through bilateral agreements, and we model their decisions as a network formation game. A financial network that allows losses to be shared among various counterparties emerges endogenously. We show that in an equilibrium network the degree of systemic risk, defined as the probability that contagion occurs conditional on one bank failing, is significantly reduced. In certain equilibria, contagion does not occur.

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1 Introduction

The ongoing turmoil in financial markets has revealed, once again, the intertwined nature of financial systems. While the events unfolded it became clear that the consequences of such an interconnected system are hard to predict. In a modern financial world, banks and other institutions are linked in a variety of ways. The incentives for linking are driven by the various benefits these connections bring. Frequently, banks solve their liquidity imbalances simply by transferring funds from the ones that have a cash surplus to those with a cash deficit. In the corporate loan market, banks often prefer syndicating loans with other banks over being the sole lender. Similarly, in the primary equity and bond markets, banks tend to co-underwrite securities offerings with banks with which they have long-standing relationships. More than anywhere else, the system's interconnectedness grew beyond recognition in the markets for credit derivatives.

In this paper we introduce a different rationale for linkages in the banking system. We explore whether banks can commit ex-ante to mutually insure each other, when the failure of an institution introduces the risk of contagion in the financial system. As a form of insurance banks can hold mutual claims on one another. The mutual claims are implemented as bilateral agreements that allow losses to be shared among all counterparties of a defaulted bank. We model banks' decision to share the risk of contagion bilaterally as a network formation game. Since links between banks allow losses to be shared among various counterparties, the financial network provides, thus, an insurance mechanism that resembles a clearing house.

Increasing the connectivity level may seem to rather enhance the sensitivity of the system to contagion. Although counter-intuitive, the idea that an interconnected banking system may be optimal is supported by various other studies. Leitner (2005) discusses how the threat of contagion may be part of an optimal network design. His model predicts that it is optimal for some agents to bail out other agents, in order to prevent the collapse of the whole network. This form of insurance can also emerge endogenously, and we show that it is an equilibrium in a network formation game. Linkages between banks can also be efficient in the model of Kahn and Santos (2008) if there is sufficient liquidity in the financial system.

We study the network formation process using a framework proposed in Allen and Gale (2000). Allen and Gale (2000) study how the banking system responds to contagion when banks are connected under different network structures. In a setting where consumers have random liquidity needs as in Diamond and Dybvig (1983), banks can perfectly insure against liquidity shocks by exchanging interbank deposits. The connections created by swapping deposits expose, however, the system to contagion. The authors show that incomplete networks are more prone to contagion than complete structures.

In our model, banks form bilateral connections to reduce the risk of contagion induced in the system through linkages that insure against liquidity shocks. We distinguish between a *liquidity network*, that smoothes out liquidity shocks in the banking system, and a *solvency network*, that is formed endogenously as insurance against contagion risk. At the base of the link formation process lies the same intuition developed in Allen and Gale (2000): better connected networks are more resilient to contagion. Our model predicts a connectivity threshold above which contagion does not occur. Thus, in order to insure against the risk of contagion, banks form links to reach this threshold. However, an implicit cost associated to being involved in a link prevents banks from forming connections more than required by the connectivity threshold. We show that banks succeed in forming networks highly resilient to the propagation of shocks.

The framework is stylized: the banking system consists of two identically sized regions, liquidity shocks are negatively correlated across these regions and there is no aggregate uncertainty. Although seemingly restrictive, these assumptions provide, in essence, microfoundations for an interconnected banking system. Their role is simply to motivate linkages driven by liquidity needs that constantly emerge between banks.[1] Indeed, the main role of interbank markets is to redistribute liquidity in the financial system from the banks that have cash in excess to the ones that have a shortage. Banks interact on this market creating linkages that are potentially conducive to contagion.

Once we have reproduced the interbank market environment, we formulate the network formation game where the incentives for linking are driven by contagion risk. The two types of interactions, for insurance against liquidity shocks and for insurance against contagion risk can thus be regarded as independent issues. This implies that for any pattern of interconnections that insures banks against liquidity shocks, we can solve the network formation game. Studying the network formation process in less symmetrical

[1] Although currently interbank markets experience a freeze, this behavior is a rather new feature of the ongoing crisis. In less agitated times, banks interact actively in these markets.

settings although feasible, would merely add complexity without adding much insight to the analysis.

It is important to emphasize the distinction between linkages that insure against liquidity risk and linkages that insure against contagion risk. The latter will decrease the risk of contagion that the former induce in the system. The immediate implication is that insuring against liquidity risk is worth the trade-off of exposing the system to a small risk of contagion. However, a large probability of contagion may offset the benefits from insuring against liquidity shocks by forming linkages through the interbank market. This may explain why we have witnessed liquidity drying up from interbank markets, as banks refused to lend money to each other.

This paper contributes to the finance literature by introducing formal networks techniques to analyze financial systems. The theory of networks may provide a conceptual framework within which the various patterns of connections can be described and analyzed in a meaningful way. The general concept of a network is quite intuitive: a network describes a collection of nodes and the links between them. The intricate structure of linkages between financial institutions can be, thus, naturally captured by using a network representation of financial systems.

A network approach to financial systems is particularly important for assessing financial stability and can be instrumental in capturing what externalities the risk associated with a single institution may create for the entire system. A better understanding of network externalities may facilitate the adoption of a macro-prudential framework for financial supervision. Regulations that target individual institutions, as well as take into account vulnerabilities that emerge from network interdependencies in the financial system may prevent a local crisis becoming global.

Situations, such as the one we study, where agents form or severe connections depending on the benefits they bring are modeled through a game of network formation. A recent and rapidly growing literature on network formation games has developed in the past few years, introducing various approaches to model network formation and proposing several equilibrium concepts (Jackson and Wolinsky, 1996, Goyal and Vega-Redondo, 2007, Bloch and Jackson, 2007, Dutta et. al, 2005).

Although there are numerous applications of these models in the social science context, the research on financial networks is still at an early stage. Allen and Babus (2008) provide a comprehensive survey of this literature. We highlight below several relevant contributions.

Most of the research done in financial networks studies network effects rather than network formation. That is, the interest is on how different network structures respond to the breakdown of a single bank in order to identify which ones are more fragile. For instance, Freixas et al. (2000) considers the case of banks that face liquidity needs as consumers are uncertain about where they are to consume. The authors analyze different market structures and find that a system of credit lines, while it reduces the cost of holding liquidity, makes the banking sector prone to experience gridlocks, even when all banks are solvent. Dasgupta (2004) also discusses how linkages between banks represented by crossholdings of deposits can be a source of contagious breakdowns. Fragility arises when depositors, that receive a private signal about banks' fundamentals, may wish to withdraw their deposits if they believe that enough other depositors will do the same. Corbae and Duffy (2008) provide an experimental investigation on how the network structure influences the decisions financial institutions take when faced with idiosyncratic risk and the possibility of contagions. Other researchers apply network techniques developed in mathematics and theoretical physics to study contagion, such as Eisenberg and Noe (2001) or Minguez-Afonso and Shin (2007). Contagion via indirect linkages on contagion is considered as well in another set of studies. Lagunoff and Schreft (2001) construct a model where agents are linked in the sense that the return on an agent's portfolio depends on the portfolio allocations of other agents. Similarly, Cifuentes et al. (2005) present a model where financial institutions are connected via portfolio holdings. Complementary to this literature, Castiglionesi and Navarro (2007) study whether banks decentralize the network structure proposed by a social planner, in a setting where banks that are not sufficiently capitalized gamble with depositors' money.

Besides the theoretical investigations, empirical studies of national banking systems have looked for evidence of contagious failures of financial institutions resulting from the mutual claims they have on one another. Most of these papers use balance sheet information to estimate bilateral credit relationships for different banking systems. Subsequently, the stability of the interbank market is tested by simulating the breakdown of a single bank. Upper and Worms (2004) analyze the banking system in Germany, Furfine (2003) the US, Wells (2004) the UK, , Boss et al. (2004) Austria, and Degryse and Nguyen (2007)

Belgium. These papers find that the banking systems demonstrate high resilience, even to large shocks. Simulations of the worst case scenarios show that banks representing less than 5% of total balance sheet assets would be affected by contagion on the Belgian interbank market, while for the German system the failure of a single bank could lead to the breakdown of up to 15% of the banking sector based on assets. For most countries, data is extracted from banks' balance sheets, which can provide information on the aggregate exposure of the reporting institution vis-a-vis all other banks. To estimate bank-to-bank exposures, it is generally assumed that banks spread their lending as evenly as possible. In effect, this assumption requires that banks are connected in a complete network. Hence the assumption might bias the results, in the light of the theoretical findings that better connected networks are more resilient to the propagation of shocks.

This paper is organized as follows. Section 2 introduces the basic model. The rationale for a liquidity network is presented in Section 3, while Section 4 describes how contagion may occur. Section 5 discusses the payoffs banks have from forming links, and models the network formation game. Section 6 analyses the efficiency of banks' link formation decisions. Section 7 concludes.

2 The Model

In this section we review the basic framework of Allen and Gale (2000) and lay the premises of our network formation game. Despite stylized assumptions, the model captures well the nature of interactions in the interbank market. In particular, banks use the interbank market to develop relationships that allow them to insure against liquidity risk, as shown empirically by Cocco et al. (2008). The aim in this section is to provide microfoundations for the role interbank markets in redistributing liquidity in the financial system.

2.1 Consumers and Liquidity Preferences

The economy is divided into $2n$ sectors, each populated by a continuum of risk averse consumers. There are three time periods $t = 0, 1, 2$. Each agent has an endowment equal to one unit of consumption good at date $t = 0$. Agents are uncertain about their liquidity preferences: with probability q they are early consumers, who value consumption only at date 1, and with probability $(1 - q)$ they are late consumers, who value consumption only at date 2. While in the aggregate there is no uncertainty about the liquidity demand in period 1, each sector experiences random fluctuations in the need for liquidity of early consumers. With probability $(1-\pi)/2$, in each sector there is either a high proportion p_H of agents that need to consume at date 1 or a low proportion p_L of agents that value consumption in period 1. Only with a small probability π, the fraction of early consumers is the same across sectors, $q = \frac{p_H + p_L}{2}$.

All the uncertainty is resolved at date 1, when the state of the world is realized and commonly known. At date 2, the fraction of late consumers in each region will be $(1-p)$ where the value of p is known at date 1 as either p_H, p_L or q.

2.2 Banks and Investment Opportunities

In each sector i there is a competitive representative bank. Agents deposit their endowment in their sector's bank. In exchange, they receive a deposit contract that guarantees them an amount of consumption depending on the date they choose to withdraw their deposits. In particular, the deposit contract specifies that if they withdraw at date 1, they receive $C_1 > 1$, and if they withdraw at date 2, they receive $C_2 > C_1$.

Banks have two investment opportunities: a liquid asset with a return of 1 after one period, or an illiquid asset that pays a return of $r < 1$ after one period, or $R > 1$ after two periods. Let x and y be the per capita amounts invested in the liquid and illiquid asset, respectively. Banks choose investments in liquid and illiquid assets such that to maximize the ex-ante expected utility of consumers

$$Eu(C_1, C_2) = qu(C_1) + (1-q)u(C_2)$$

Since banks can transfer utility from late consumers to early consumers, depositors find the deposit contract beneficial.

The uncertainty in the liquidity preferences of consumers is summarized in Table 1. We assume that early and late consumers are distributed across banks such that two regions, A and B, are created in the banking system. In each region there is a equal number of banks: $A = \{1, 2, ..., n\}$ and $B = \{n+1, n+2, ..., 2n\}$. Moreover, fluctuations in the fraction of early consumers are negatively correlated across regions. Hence, when banks in region A receive a high fraction, for instance, banks in region B receive a low fraction, and the other way around.

		Region A				Region B			
Probability	State/Bank	1	2	...	n	$n+1$	$n+2$	...	$2n$
$(1-\pi)/2$	S_1	p_H	p_H	...	p_H	p_L	p_L	...	p_L
$(1-\pi)/2$	S_2	p_L	p_L	...	p_L	p_H	p_H	...	p_H
π	$\bar{S}$	q	q	...	q	q	q	...	q

Table 1: Distribution of Shocks in the Banking System

3 The Liquidity Network and Optimal Risk Sharing

The optimal risk sharing problem is well understood when there is no aggregate uncertainty about the fraction of early consumers. Allen and Gale (2000) characterize optimal risk sharing as the solution to a planning problem. They show that the planner's portfolio investment decision allocates the liquid asset to pay the early consumers and the illiquid asset to pay the late consumers. In other words, the portfolio allocation between the liquid and illiquid asset that maximizes the expected utility of consumers is

$$(x, y) = (qC_1, (1-q)C_2)$$

An interbank market can decentralize the planner's solution as long as it allows banks that have a high fraction of early witdrawals to raise the liquidity they need from the banks that have a low fraction of early withdrawals. This is possible since there is no aggregate uncertainty and each bank at date has either a liquidity surplus of $zC_1 = (q - p_L)C_1$, a liquidity shortage of $zC_1 = (p_H - q)C_1$ or can perfectly meet withdrawals from early consumers, depending on the state of the world realized.

Negatively correlated liquidity shocks across the two regions create opportunities for risk-sharing. Banks can hedge completely individual risk by exchanging interbank deposits between banks in different regions, at date 0.[2] We assume each bank receives the same return as the consumers for the amounts transferred as deposits: C_1, if they withdraw after one period, and C_2 if they withdraw after two periods. These interactions create balance sheet linkages between banks in the two regions.

[2]Exchanging interbank deposits *ex-ante* in order to insure against liquidity shocks may be seen as uncoventional. Acharya et al. (2008) emphasize some of the problems that occurs when liquidity transfers occur *ex-post*. For instance, surplus banks may strategically under-provide lending to induce inefficient sales of assets from needy banks.

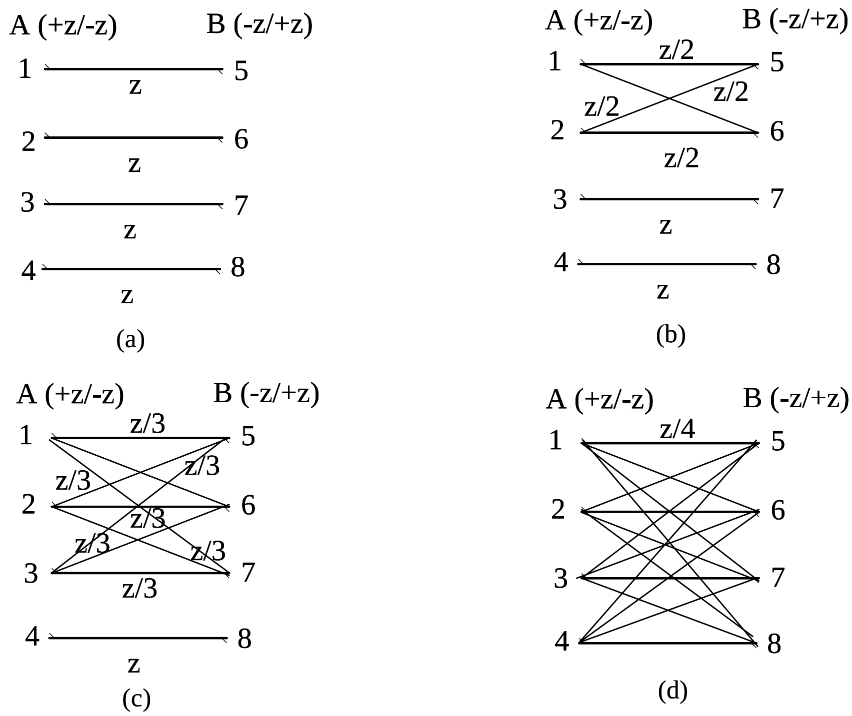

Figure 3.1: Interbank market connections

For the simplicity of exposition, we omit the calculations that show why an interbank market is risk-sharing optimal.[3] We want to stress though that any pattern of connections between banks in different regions can decentralize the planner's solution. Figure 3.1 shows various ways through which bank can perfectly insure against liquidity shocks, where the linkages represent deposits exchanged at date 0.

Thus, banks' portfolios consist of three assets: the liquid asset, the illiquid asset and the interbank deposits. Each of these three assets can be liquidated in either of the last 2 periods. However, the costliest in terms of early liquidation is the illiquid asset, followed by interbank deposits. This implies the following ordering of returns:

$$1 < \frac{C_2}{C_1} < \frac{R}{r} \tag{3.1}$$

Let a_{ij} denote the amount exchanged as deposits between banks i and j at date 0. We consider that deposit contracts are bilateral, hence we have $a_{ij} = a_{ji}$. Let N_i be the set of banks i is linked to and let N_i^{cross} be a subset of N_i representing the banks i is linked

[3] Allen and Gale (2000) show in detail why this is the case.

to in the other region. Then, the total amount of deposits i exchanges with its neighbors should balance out its liquidity shortage or excess. Since the insurance against liquidity shocks is provided only through links with banks in a different region, a_{ij} should satisfy the *feasibility constraint*:

$$\sum_{j \in N_i^{cross}} a_{ij} \geq z \tag{3.2}$$

4 Contagion Risk

The scope in this paper is to study whether banks can commit ex-ante to mutually insure each other, when there is a risk of contagion in the financial system. To analyze this question, we allow any bank to default in state $\bar{S}$. The default is modelled as a non-insurable idiosyncratic shock. When such event is introduced, the probability of a crisis becomes positive, and optimal risk sharing may no longer be achieved at date 0. As Allen and Gale (2000) point out, the feasible actions for a bank are difficult to characterize when there is aggregate uncertainty. For instance, banks may decide to invest more in the liquid asset and rely less on the interbank market. The approach we take in this paper is to give banks the possibility to mutually insure against the risk of contagion.

We are, thus, interested in understanding whether banks internalize the negative externality that contagion may create in the financial system. To allow the idiosyncratic shock to have a negative externality on other banks in the system, we assume that banks are locked in the (C_1, C_2) contract with depositors. That is, the investment in the liquid and illiquid asset is fixed as before at $(x, y) = (qC_1, (1-q)C_2)$. When banks are forced to rely on the interbank market for smoothing liquidity shocks of zC_1, the idiosyncratic shock that initially affects only one institution can propagate through the entire system. In this section we explain how such shock spills over from one institution to another. We model how banks respond to it by committing to mutually insure each other in the following section.

4.1 Losses Given Default

As a measure to evaluate contagion risk we use *loss given default* (henceforth LGD). LGD expresses the excess of nominal liabilities over the value of the assets of the failed bank.

In our setting, LGD will be given by the loss of value a bank incurs on its deposits when one of its neighbor banks is liquidated.

To calculate LGD, we need to determine the value of the assets of the failed bank. When a bank i fails, its portfolio of assets is liquidated at the current value and distributed equally among creditors. The three assets in banks' portfolio yield different returns upon liquidation in period 1. First, the amount of x per capita banks invested in the liquid asset pays a return of 1. Second, the amount y per capita banks hold in the illiquid asset, pays a return of $r < 1$ if liquidated early. And lastly, the interbank deposits, summing up to $\sum_{k \in N_i} a_{ik}$, yield a return of C_1 per unit of deposit. On the liability side, a bank has to pay its depositors, normalized to 1 and at the same time to repay its interbank creditors that also add up to $\sum_{k \in N_i} a_{ik}$. This yields a new return per unit of good deposited in a bank i equal to $\bar{C}_i = \frac{x+ry+\sum_{k \in N_i} a_{ik} C_1}{1+\sum_{k \in N_i} a_{ik}} < C_1$.[4] The LGD of bank j given that bank i has failed is expressed as[5]:

$$LGD_{ji} = a_{ji}(C_1 - \bar{C}_i) = a_{ji} \frac{C_1 - x - ry}{1 + \sum_{k \in N_i} a_{ik}} \tag{4.1}$$

It is clear that the loss given default between any pair ij is increasing in amount of deposits a_{ji} exchanged between the two banks. A second, less intuitive implication follows from the expression for LGD in eq. (4.1). That is, everything else equal, the more connected one bank is, the smaller the loss it induces to its neighbors in case it fails. In other words, the more connected one bank's neighbors are, the better the respective bank is.

4.2 Contagion Threshold

We describe below the contagion mechanism. When a bank fails, its neighbors incur a loss on the value of deposits exchanged with the failed bank. This implies that, in period 1, affected banks hold the value of the liquid asset, qC_1, less the size of LGD. Hence, to meet its obligation, qC_1, towards early consumers, a bank oughts to liquidate an amount of the illiquid asset that equals the LGD value. Liquidating the illiquid asset prematurely, however, involves a penalty rate $r < 1$ and has negative consequences for the late consumers. In fact, if too much of the illiquid asset is liquidated early, the consumption of

[4] Eq. (3.1) ensures that the inequality holds.

[5] In principle $LGD_{ji} \neq LGD_{ij}$ since it may be that $\sum_{k \in N_i} a_{ik} \neq \sum_{k \in N_k} a_{jk}$

late consumers may be reduced to a level below C_1. In this case, the late consumers gain more by imitating the early consumers and withdrawing their investment from the bank at date 1. This induces a run on the bank and, subsequently, triggers its failure.

The maximum amount of illiquid asset that can be liquidated without causing a run depends on the fraction of late consumers, $(1-q)$, and the return rates for early and late liquidation of the illiquid asset, r and R. Equation (4.2) captures the exact effect of these variables.

$$b(q) \equiv r\left[y - \frac{(1-q)C_1}{R}\right] \tag{4.2}$$

The maximum amount of illiquid asset that can be liquidated without causing a run on the bank can be interpreted as a *contagion threshold*. Any bank that incurs a LGD higher than $b(q)$ will, inevitably, fail. A value of LGD below the threshold $b(q)$ will not trigger the failure of a bank. However, it will be costly for the late consumers, given that their consumption is now reduced to $\tilde{C}_2 < C_2$.[6] There is, thus, an implicit cost associated with being involved in a link. Links are potentially conduits of LGD, which is detrimental for banks that incur it, even when below the contagion threshold $b(q)$.

5 The Solvency Network and Mutual Insurance

While the idiosyncratic shock in state $\bar{S}$ is not insurable, the risk of contagion may be reduced if banks choose to mutually insure each other. As a form of insurance, banks can hold mutual claims on one another. The mutual claims constitute bilateral agreements that allow losses to be shared among all counterparties of the defaulted bank. We model banks' decision to commit ex-ante to mutually insure as a network formation game, where banks' actions are driven by the risk of contagion.

The risk of contagion changes the expected utility of consumers, and hence, the optimization problem of banks. In state $\bar{S}$, consumers may expect a consumption vector of $(C_1, \tilde{C}_2)$ or $(\bar{C}_1, \bar{C}_1)$, depending on whether the LGD is above or below the contagion threshold $b(q)$. By mutually insuring each other and sharing losses in state $\bar{S}$, banks can improve the utility of consumers from $Eu(\bar{C}_1, \bar{C}_1)$ to $Eu(C_1, \tilde{C}_2)$.

We study how banks insure against contagion risk when the liquidity network is complete. This implies that each bank in one region is linked with all banks in the other region,

[6]The consumption of late consumers at least equals the consumption of the early consumers: $\tilde{C}_2 \geq C_1$.

as in fig. 3.1 (d), and the amount of deposits exchanged between any two banks is z/n. While we can model the network formation process for any given pattern of interactions between banks across regions, we choose the complete liquidity network as a benchmark for simplicity. Moreover, the amount of deposits exchanged per link is minimal when each bank in one region is linked to all the other banks in the other region (Lemma 1 in the appendix formalizes this result). While such an interconnected liquidity network seems to exacerbate systemic risk, recall that contagion depends on the value of LGD. This implies that if the idiosyncratic shock triggers contagion in the complete liquidity network, then it will also trigger contagion in less dense networks, as long as there exists an indirect connection between any two banks in the system.

Thus, we investigate whether banks can endogenously reduce contagion risk, by forming links with other banks in the same region. Links across regions provide perfect insurance against liquidity shocks and they connect banks in a liquidity network. Links with banks in the same region connect the bank in a solvency network.

5.1 Concepts and Notations

Let $N = \{1, 2, ..., 2n\}$ denote the set of banks. A network g on the set N is a collection of g_{ij} pairs, with the interpretation that i and j are linked. Thus, if i and j are linked in the network g, then $g_{ij} \in g$.

The set of *neighbors* of bank i in the network g is $N_i(g) = \{j \in N \mid g_{ij} \in g\}$. Let $\eta_i(g) = |N_i(g)|$, where $|\cdot|$ represents the cardinality of a finite set. The number of neighbors, $\eta_i(g)$, bank i has in the network g is called the *degree* of bank i. In addition, let $N_i^{inner}(g) = \{j \in N \,| g_{ij} \in g \text{ and } i, j \in A \text{ or } i, j \in B\}$ and $\eta_i^{inner}(g) = \left|N_i^{inner}(g)\right|$. The number of neighbor banks in the same region is defined as the *inner degree* of bank i. A related notation is used for the set of banks in a different region, $N_i^{cross}(g) = N_i(g) \setminus N_i^{inner}(g)$, and the number of neighbor banks in a different region is $\eta_i^{cross}(g) = \left|N_i(g) \setminus N_i^{inner}(g)\right|$.

We use the notation $g + g_{ij}$ to denote the new graph obtained from g by linking i and j, if $g_{ij} \notin g$. Similarly, we consider that $g - g_{ij}$ represents the graph obtained from g by deleting an existent link between i and j, when $g_{ij} \in g$

A path of length k between i and j is a sequence of distinct agents $(i, j_1, ..., j_{k-1}, j)$ such that g_{ij_1}, $g_{j_1 j_2}$,..., $g_{j_{k-1}j} \in g$. A network g is connected if there exists a path between any two nodes i and j from N. A network g is complete if for any node $i \in N$, $\eta_i(g) = n - 1$.

A network g is regular of degree k if for any node $i \in N$, $\eta_i(g) = k$.

5.2 Strategies and Incentives

We model the interaction between banks in the same region as a network formation game. The solvency network is formed as a result of banks' actions, who decide how to form links. In this case, links are represented as well by deposits that can be withdrawn upon demand. For simplicity, we normalize the amount of deposits exchanged between banks i and j in the same region to $a_{ij} = \frac{z}{n}$. In essence, these mutual claims on one another allow banks to share losses in state $\bar{S}$.

Deposits are exchanged on bilateral basis (i.e. bank i agrees to pass its deposits to bank j if and only if bank j will pass its deposits to bank i in turn), which implies that the network is undirected and the formation of a link requires the consent of both parties involved. However, the severance of a link can be done unilaterally, as deposits can be withdrawn on demand by either of the banks involved in a link (in this case, deposits will be restituted to both banks, although only one party exercises the claim).

The strategy of bank $i \in A$ (B) can be described as a linking vector $s_i = (s_{i1}, s_{i2}, ..., s_{in})$ such that $s_{ij} \in \{0, 1\}$ for each $j \in A \setminus \{i\}$ $(B \setminus \{i\})$ and $s_{ii} = 0$, where $s_{ij} = 1$ means that i intends to form a link with bank j. A link between i and j is formed if and only if $s_{ij} = s_{ji} = 1$.[7] Links formed between banks in the same region serve solely to limit contagion.

We briefly give the intuition behind the payoffs banks gain from linking to other banks. The *contagion threshold* we introduced in Section 3.3 is useful to describe the trade-offs of the linking process. When a failed bank induces a value of LGD higher than the contagion threshold, $b(q)$, then all the the neighboring banks will fail in turn. This chain of failure will affect remaining banks, causing, in the end, the failure of the entire system. Note that the propagation magnitude of the idiosyncratic shock is independent on the number of neighbors of the bank affected initially. If there exists a connecting path between any two banks in the system, contagious failure will always occur when $LGD \geq b(q)$. However, if the LGD caused by a failed bank is below the threshold $b(q)$, only the neighboring banks experience a loss. This reduces the consumption of the late consumers to a level $\tilde{C}_2 < C_2$,

[7] This condition capture that the formation of a link between two banks requires the consent of both participants.

as explained in Section 3.5. Any other bank is able to pay C_1 to the early consumers and C_2 to the late consumers.

Whether LGD is above or below the contagion threshold, $b(q)$, depends on how well connected the respective bank is. The size of deposits exchanged between banks in different regions is z/n. The size of deposits exchanged between banks in different regions is also z/n. This implies that the LGD induced by a failed bank depends on how many neighbors the respective bank has.

The contagion threshold, $b(q)$, is identical for all banks and is independent of the number of links a bank has. Thus we can identify a number $t \in \mathbb{N}$ that brackets $b(q)$ as follows:

$$\frac{z}{n}\frac{C_1 - x - ry}{1 + (n+t)\frac{z}{n}} \leq b(q) < \frac{z}{n}\frac{C_1 - x - ry}{1 + (n+t-1)\frac{z}{n}} \tag{5.1}$$

The left hand side of the inequality is exactly the LGD induced by a bank with $(n+t)$ neighbors, and the right hand side is the LGD induced by a bank with $(n+t-1)$ neighbors. Inequality 5.1 relates the contagious effects of a bank failure to the number of links banks have. For a given value of the contagion threshold $b(q)$, the failure of a bank with $(n+t-1)$ links or less triggers the failure of the entire system, through the mechanism described above. The failure of a bank with at least $(n+t)$ links, however, affects only neighbor banks, which incur a loss.

We discuss in detail the implications of a bank failure when $t \in \{1, 2, ..., n-1\}$. Consider the failure of a bank j in a network g. Bank j is assumed to be linked to all the banks in a different region, which implies $\eta_j^{cross} = n$. We distinguish the following cases:

1. $\eta_j^{inner}(g) < t$. In this case, for any $i \in N_j(g)$ we have $LGD_{ij} > b(q)$. Consequently, any bank $k \in N$ will also fail.[8]

2. $\eta_j^{inner}(g) \geq t$. In this case, for any $i \in N_j(g)$ we have $LGD_{ij} \leq b(q)$. Thus, any bank $i \in N_j(g)$ will pay the early consumers C_1. The late consumers, however, will have their consumption reduced to $\tilde{C}_2 < C_2$. Any other non-neighboring bank $k \in N \setminus N_j(g)$ will not be affected in any way and will be able to pay its consumers C_1 at date 1 and C_2 at date 2.

[8] With this discussion, we are able to conclude that the existence of a single bank with insufficient link may trigger the failure of the entire system.

5.3 Payoffs

We formally introduce the payoff a bank i gains from network g. Banks seek to minimize the probability of failure through contagion. This creates incentives to form links in order to bring LGD below the contagion threshold, $b(q)$. However, there is an implicit cost of linking: losses are transmitted through links, even when $LGD \leq b(q)$.

Let $b(q)$ be the contagion threshold and t an integer that satisfies 5.1. Formally, we can express the payoff of a bank $i \in N$ as a function u, increasing in the number of nodes with an inner degree higher than t and decreasing in the number of neighbors with an inner degree higher than t.

$$u_i(g) = f(|T|, |N_i(g) \cap T|) \tag{5.2}$$

where $T = \{j \in N \left| \eta_j^{inner}(g) \geq t\right.\}$ and $|\cdot|$ represents the cardinal of a set.

Moreover, for any node i and any inner degree η_i^{inner}, the payoff u_i has the following *properties*:

1. $u_i(g + g_{ij}) = u_i(g)$ and $u_i(g - g_{ij}) = u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner}(g) < t - 1$

   The explanation for this indifference relies on the fact that the failure of a node with an inner degree below t will trigger the failure of the entire system. The failure of j leads to the failure of i, regardless of i creating a link or severing an existent link with j.

2. $u_i(g + g_{ij}) > u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner}(g) = t - 1$

   If $\eta_j^{inner}(g) = t - 1$ and i creates a link with j, then the inner degree of j becomes $\eta_j^{inner}(g) = t$. Thus, i trades a situation when the failure of j induces its own failure, for a situation when the failure of j results in merely a lower utility for i's late consumers.

3. $u_i(g + g_{ij}) < u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner}(g) \geq t$

   When j has already an inner degree sufficiently high, its failure will have only effects for the neighbor banks. Linking with j does not bring i any benefits, but it comes at the cost represented by the loss i might incur if j fails.

4. $u_i(g - g_{ij}) > u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner}(g) \geq t + 1$

   Severing an existent link with j, will leave j with an inner degree still sufficiently high. It will, however, spare i from experiencing a loss in case j fails.

5.4 Equilibrium Networks

To identify equilibrium networks, we use the following concept introduced by Jackson and Wolinsky (1996).

Criterion 1 *Let* $g_{ij} = \min(s_{ij}, s_{ji})$ *and consider that* $g_{ij} \in g$ *when* $g_{ij} = 1$*. A network* g *is pairwise stable if*

1. *for all* $g_{ij} \in g$, $u_i(g) \geq u_i(g - g_{ij})$ *and* $u_j(g) \geq u_j(g - g_{ij})$ *and*

2. *for all* $g_{ij} \notin g$, *if* $u_i(g) < u_i(g + g_{ij})$ *then* $u_j(g) > u_j(g + g_{ij})$.

where $u_i(g)$ *is the payoff of bank* i *in the network* g.

The first condition of the stability criterion states that a network is stable if there is no bank that wishes to severe a link it is involved in. The second condition requires that in a stable network there are no two unconnected banks that would both benefit by forming a link. In other words, a network is stable if there are no banks that wish to deviate either unilaterally (by severing existent links), or bilaterally (by adding a link between two banks).

The first result provides a necessary condition for a stable network to exist.[9]

Proposition 1 *Let* $b(q)$ *be the contagion threshold and* t *an integer that satisfies 5.1. If a network* g *is pairwise stable, then any bank* $i \in N$ *has an inner degree* $\eta_i^{inner}(g) \leq t$.

Proof. The proof follows immediately from properties 3 and 4 described in the previous section, properties that characterize the payoffs a bank i gains from the network g. Suppose that there exists a bank i such that $\eta_i^{inner}(g) > t$. Than, any neighbor $j \in N_i(g)$ has an incentive to severe the link that connects it with i. This way, g is no longer stable. ■

This result provides only a partial characterization of stable networks. In fact, under payoffs that respect properties $1 - 4$, there many pairwise stable networks. In fact, any network where each node i has an inner degree $\eta_i^{inner}(g) \leq t - 2$ is pairwise stable. This multiplicity of equilibria is mainly driven by the indifference in forming or severing links expressed by property 1. In what follows we alter property 1 in order to restrict the set of

[9] All results in this section hold under the assumption that the crossing degree of any bank $i \in N$, in the network g, is $\eta_i^{cross}(g) = n$.

stable networks. Namely, we consider that banks have a weak preference to forming links with other banks. We keep the other properties as described in Section 4.4.

Formally, if $b(q)$ is the contagion threshold and t is an integer that satisfies the inequality 5.1, then for any node i and any inner degree η_i^{inner}, the payoff u_i has the following *properties*:

1'. $u_i(g + g_{ij}) = u_i(g) + \varepsilon$ and $u_i(g - g_{ij}) = u_i(g) - \varepsilon$, $\forall j \in N$ s.t. $\eta_j^{inner} < t - 1$;

2. $u_i(g + g_{ij}) > u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner} = t - 1$;

3. $u_i(g + g_{ij}) < u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner} \geq t$;

4. $u_i(g - g_{ij}) > u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner} \geq t + 1$.

Under this new set of properties we can comprehensively characterize the set of stable networks and make a prediction about the stability of the banking system. Identifying stable networks architectures is not a scope as such, but rather evaluating their resilience to contagion. Contagion stems from banks that have insufficient links, and a bank i has insufficient links when $\eta_i^{inner} < t$. Hence, we aim to characterize stable networks depending on the number of banks that have at least t neighbors in the same region ($\eta_i^{inner} = t$). The following two results provide necessary conditions for the existence of stable networks.

Proposition 2 *Let g be a pairwise stable network and $T = \{i \in N \mid \eta_i^{inner}(g) = t\}$. Then $|T| \geq 2(n - t)$.*

Proof. The proof is provided in the Appendix. ■

Proposition 2 has strong implications for the stability of the banking system, especially when t is small. If t is small, proposition 2 shows that in equilibrium most of the banks have sufficient links to prevent a shock in one of the institutions spreading through contagion. For t large, however, the predictions are weaker. The following proposition provides such a refinement for large values of t..

Proposition 3 *Let g be a pairwise stable network and $T = \{i \in N \mid \eta_i^{inner}(g) = t\}$. If $t \geq n/2$, then $|T| \geq n$.*

Proof. The proof is provided in the Appendix. ■

Proposition 3 states that in a stable network, at least half the banks will have a sufficiently large number of links such that the losses they may generate are small enough.

The contagion threshold, $b(q)$, depends positively in the return rate for early liquidation of the illiquid asset, r. Since t is the smallest integer such that $b(q) < \frac{z}{n}\frac{C_1 - x - ry}{1+(n+t-1)\frac{z}{n}}$, a low value of the contagion threshold requires a large t. Hence, when the return rate for early liquidation if the illiquid asset, r, is low, banks need a large number of connections with banks in the same region, for the LGD to be below the contagion threshold. Similarly, a high opportunity cost of liquidating early the illiquid asset, $\frac{R}{r}$, decreases the contagion threshold and increases the number of links a bank requires not to be a source of contagion.

The following two results relate these findings to implications for the stability of the system.

Corollary 1 *Let g be a pairwise stable network and $T = \{i \in N \mid \eta_i^{inner}(g) = t\}$. If $t < n/2$, then the probability that the failure of a bank will spread through contagion is at most* $\min(t\pi/n, \pi/2)$.

Proof. The proof follows simply from Propositions 2 and 3. ■

This result implicitly insures that for high levels of the contagion threshold the probability of contagion is significantly low. The intuition for this result relies on the fact that the higher the contagion threshold is, the lower the number of links banks need in order prevent contagion. Proposition 2 indicates that a lower connectivity of the banking system is easier to obtain. A low level of the contagion threshold, however, requires a high connectivity in the banking system. Hence, the probability of contagion is higher.

6 Inefficiencies and Systemic Risk

The degree of systemic risk, defined as the probability that contagion occurs conditional on one bank failing, is determined endogenously. Banks choose to form links as a form of insurance against contagion. However, the networks that result from such choices are still exposed to the risk that the initial shock spreads through contagion. The probability of a systemic failure is determined endogenously, as a consequence of banks' strategic actions.

A social planner, whose objective is to maximize the aggregate welfare of consumers, is mainly concerned about systemic risk. In our framework, systemic risk is given by the number of banks that fail, following a shock in a single institution. Minimizing the number of banks that fail through contagion reduces, thus, welfare losses. Banks' and social planner's incentives to form networks are partially aligned.

For a given contagion threshold, $b(q)$, and an integer t that satisfies 5.1, the failure of a bank with at least $(n+t)$ links does not propagate through contagion, although it affects neighbor banks that incur a loss. Thus, there is a large number of efficient networks a social planner can design in order to prevent contagion in the banking system. The set of efficient networks is characterized in the following proposition.

Proposition 4 *Let g be a network such that $\eta_i^{cross}(g) = n$ and $\eta_i^{inner}(g) \geq t$, $\forall\ i \in N$. Then g is efficient.*

Proof. Since $\eta_i^{inner} \geq t$, $\forall\ i \in N$ then it follows immediately that $\eta_i \geq n+t$, $\forall\ i \in N$. Thus, for any pair ij it must be that $LGD_{ij} \leq \frac{z}{n}\frac{C_1-x-ry}{1+(n+t)\frac{z}{n}}$. As the limit loss $b(q)$ satisfies inequality 5.1, then $LGD_{ij} \leq b(q)$ for any pair of banks ij. Hence, in the network g the failure of a bank will not trigger the failure of other banks in the system. ■

The conflict between efficient outcomes and individual incentives is a classical theme in economics. In this model, however, the incentives are partially aligned. Indeed, the set of stable networks, described by proposition 2 and 3, includes an efficient network. It is easy to check that a network g such that $\eta_i^{inner}(g) = t$, $\forall\ i \in N$ is pairwise stable and, by proposition 4, is also efficient.[10]

The set of equilibrium networks, nevertheless, incorporates many inefficient networks, especially when t is large. Inefficient equilibria arise since any bank has an incentive to free-ride on links that other banks establish. The implicit cost associated to being involved in a link, modelled as the loss one bank would face if the other one fails, limits the number of links a bank forms. In fact, if a bank can benefit from a well connected banking system without incurring the cost linking, then it will not accept forming links even when itself poses a risk for entire system (see Proposition 1).

We have characterized a particular set of equilibrium networks. Namely, we have assumed that the liquidity network is complete ($\eta_i^{cross} = n$, $\forall\ i \in N$) and we have modeled the link formation process that takes place between banks in the same region. Other equilibrium networks exist as well when the crossing degree is smaller than n. The full

[10] The existance of such an equilibrium efficient network is conditional on the existance of a t-regular network. Lovasz (1979) discusses in detail conditiond for the existance of a t-regular network with n nodes.

characterization of the set of pairwise stable networks is possible, however not of much interest: No efficient equilibrium can emerge when there exist nodes such that the crossing degree is smaller than n, for the given set of parameters.

7 Conclusions

The problem of contagion within the banking system is an intensely debated issue. This paper contributes to the existent literature by developing a model that explains how interdependencies between banks emerge endogenously. In particular, we develop a model of network formation for the banking system. We investigate how banks form links with each other, when the banking system is exposed to contagion risk. The question we address is wether banks form networks that are resilient to the propagation of small idiosyncratic shocks.

Banks respond to contagion risk by forming links. The stable network architectures that emerge are very likely to support systemic stability. For instance, when the probability of a idiosyncratic shock is π, then the probability that it will spread by contagion is at most $\pi/2$. Moreover, in certain equilibria contagion does not occur.

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A Appendix

In what it follows we will prove some results advanced in the main text.

Lemma 1 *Consider the states of the world* S_1 *and* S_2*, when liquidity shocks are negatively correlated across the two regions A and B. The minimization problem for LGD associated to each link, under the feasibility constrain (3.2), has a symmetric solution when each* $i \in A$ *is linked to each* $i' \in B$*, and* $a_{ii'} = \frac{z}{n}$*.*

Proof. The optimization problem is:

$$\forall\ i \in N, i' \in N_i^{cross}, \ \min_{a_{ii'}} LGD_{ii'}, \tag{A.1}$$

$$\text{s.t.} \sum_{i' \in N_i^{cross}} a_{ii'} = z \tag{A.2}$$

First we show that $LGD_{ii'}$ is decreasing in $a_{ii'}$. For this it is useful to express LGD as

$$LGD_{ii'} = a_{ii'} \frac{C_1 - x - ry}{1 + a_{ii'} + \sum_{\substack{k \in N_i(g) \\ k \neq i'}} a_{ik}} \tag{A.3}$$

The derivative of $LGD_{ii'}$ with respect to $a_{ii'}$ is given by

$$\frac{\partial LGD_{ii'}}{\partial a_{ii'}} = \frac{(C_1 - x - ry)(1 + \sum_{\substack{k \in N_i(g) \\ k \neq i'}} a_{ik})}{(1 + a_{ii'} + \sum_{\substack{k \in N_i(g) \\ k \neq i'}} a_{ik})^2} > 0 \tag{A.4}$$

A positive sign for the derivative implies that $LGD_{ii'}$ is increasing in $a_{ii'}$.

The only restriction in minimizing LGD_{ij} is the feasibility constraint (3.2). According to it, any bank i needs to insure that the amount of deposits exchanged with banks of a different type sums up to z.

We impose that the solution is symmetric. That is $a_{ii'} = \frac{z}{\gamma}$. Since there are n banks of a different type and $LGD_{ii'}$ is increasing in the amount of deposits a_{ij}, the solution to the minimization problem dictates that each bank creates links to all the other banks of a different type. Subsequently, the amount exchanged on each link is $a_{ij} = \frac{z}{n}$. ■

Proposition 1 *Let g be a pairwise stable network and* $T = \{i \in N \left| \eta_i^{inner}(g) = t \right.\}$*. Then* $|T| \geq 2(n - t)$*.*

Proof. Consider the two regions in the banking system $A = \{1, 2, ..., n\}$ and $B = \{n + 1, n = 2, ..., 2n\}$. Let $T(A) = \{i \in A \left| \eta_i^{inner}(g) = t \right.\}$ and $T(B) = \{i \in B \left| \eta_i^{inner}(g) = t \right.\}$. Clearly we have $|T| = |T(A)| + |T(B)|$. In order to prove that $|T| \geq 2(n - t)$, we show that $|T(A)| \geq n-t$ and $|T(B)| \geq n-t$. Since the cases are symmetric, we prove only that $|T(A)| \geq n-t$. For this we assume the contrary in order to arrive to a contradiction.

Suppose that $|T(A)| < n-t$. This implies that the set $T(A)$ has at most $n-t-1$ elements. Further, this implies that $|A - T(A)| \geq n-(n-t-1)$. In other words, the set of banks with an inner degree $\eta_i^{inner}(g) < t$ has at least $t+1$ elements. By property 1' and 2 above we know that in a stable network all the banks such that $\eta_i^{inner} \leq t-1$ must be directly linked with each other. Since the set of banks with this property is at least $t+1$, it must be that each bank in $A-T(A)$ has an inner degree $\eta_i^{inner} \geq t$. We arrived thus to a contradiction. ■

Proposition 2 *Let g be a pairwise stable network and $T = \{i \in N \,|\eta_i^{inner}(g) = t\}$. If $t \geq n/2$, then $|T| \geq n$.*

Proof. The proof follows similar steps as the proof for the previous result. Adopting the same notations, we prove only that $|T(A)| \geq n/2$.

Let $|T(A)| = \tau$. If $\tau \geq t$, the proof is complete.

Consider the case when $\tau < t$. By property 1' and 2 above we know that in a stable network all the banks in the set $A - T(A)$ must be directly linked with each other. This implies that the total number of links[11] between banks of the same type with an inner degree $\eta_i^{inner} < t$ must be $(n-\tau)(n-\tau-1)$. In addition, since $\tau < t$, it must be that each bank in $T(A)$ has some links with banks in $A - T(A)$. Assuming that all banks in $T(A)$ are directly linked with each other, there must be at least $\tau(t-\tau+1)$ links with banks in $A - T(A)$.

Since all the banks in $A - T(A)$ have an inner degree $\eta_i^{inner} < t$, the total amount of links these banks have should not exceed $(n-\tau)t$. Thus, the following inequality must hold:

$$(n-\tau)(n-\tau-1) + \tau(t-\tau+1) < (n-\tau)t$$

This inequality can be rewritten as

$$(t-\tau+1)(2\tau-n) < (n-\tau)(2\tau-n)$$

[11] Links here are counted twice for each node. However, we maintain the same double counting for the rest of the proof, such that in the end it cancels out.

Since $t - \tau + 1 < n - \tau$, it must be that $2\tau - n > 0 \Leftrightarrow \tau > n/2$. This concludes the proof. ■

Proposition 3 *Let g be a bilateral equilibrium network. Then the probability that the failure of a bank will spread through contagion is at most $\pi/2n$.*

Proof. We show that in a bilateral equilibrium networks there exists at most one node i such that $\eta_i^{inner} < t$.

Suppose that in an equilibrium network there exist at least two nodes i and j such that $\eta_i^{inner} < t$, $\eta_j^{inner} < t$. In a network that there are at least two nodes with an insufficient number of links, there are two sources of contagious failure. Thus the probability a bank associates to failing by contagion is at least $2\pi/2n$.

Let $\tilde{g}$ be such a network. Then there exist a pair ij of nodes of a different type (i.e. $i \in A$ and $j \in B$) such that it pays off to severe the links they are involved in and form the link $\tilde{g}_{ij}$, if $\tilde{g}_{ij} \notin \tilde{g}$. Formally, let $\tilde{s}_i$ and $\tilde{s}_j$ be the strategy profile bank i and bank j follow, respectively, in network $\tilde{g}$.And let $s_i^* = (\underbrace{0, 0, ..., 0}_{1,...,j-1}, 1, \underbrace{0, 0, ..., 0}_{j+1,...,n})$ and $s_j^* = (\underbrace{0, 0, ..., 0}_{1,...,i-1}, 1, \underbrace{0, 0, ..., 0}_{i+1,...,n})$. Then

$$u_i(\tilde{g}(s_i^*, s_j^*, \tilde{s}_{-i-j})) > u_i(g(\tilde{s}_i, \tilde{s}_j, \tilde{s}_{-i-j})) \text{ and } u_j(\tilde{g}(s_i^*, s_j^*, \tilde{s}_{-i-j})) > u_j(\tilde{g}(\tilde{s}_i, \tilde{s}_j, \tilde{s}_{-i-j})) \tag{A.5}$$

In the new network, the only link i and j have is $\tilde{g}_{ij}$ and thus they are exposed to contagion stemming from one source. If $\tilde{g}_{ij}$ is the only link banks i and j have, thus this link will bear the entire amount of deposits necessary to provide insurance against liquidity shocks: z. Thus, if one of the banks fails, the other one fails by necessity, since the loss it incurs is much above the limit loss threshold. However, the probability that one of the two banks will fail is $\pi/2n$ and smaller than in the network $\tilde{g}$. Hence, $\tilde{g}$ cannot be an equilibrium.

Since, in a bilateral equilibrium there exists at most one node i such that $\eta_i^{inner} < t$, it follows that the probability that the failure of a bank will spread through contagion is at most $\pi/2n$. ■