The Formation of Financial Networks

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Abstract

Modern banking systems are highly interconnected. Despite various benefits, linkages between banks carry the risk of contagion. In this paper I investigate whether banks can commit ex-ante to mutually insure each other, when there is contagion risk in the financial system. I model banks' decisions to share this risk through bilateral agreements. A financial network that allows losses to be shared among various counterparties arises endogenously. I characterize the probability of systemic risk, defined as the event that contagion occurs conditional on one bank failing, in equilibrium interbank networks. I show that there exist equilibria in which contagion does not occur.

Keywords: financial stability; network formation; contagion risk;

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1 Introduction

The recent turmoils in financial markets have revealed, once again, the intertwined nature of financial systems. In a modern financial world, banks and other institutions are linked in a variety of ways. These connections often involve trade-offs. For instance, although banks can solve their liquidity imbalances by borrowing and lending on the interbank market, they expose themselves, at the same time, to contagion risk. How do banks weigh these trade-offs, and what are the externalities of their decisions on the financial system as a whole?

In this paper I explore whether banks can commit ex-ante to mutually insure each other, when the failure of an institution introduces the risk of contagion in the financial system. As a form of insurance, banks can hold mutual claims on one another. These mutual claims are, essentially, bilateral agreements that allow losses to be shared among all counterparties of a failed bank. The more bilateral agreements a bank has, the smaller the loss that each of her counterparties incurs. Contagion does not take place provided each bank has sufficiently many bilateral agreements. I model banks' decision to share the risk of contagion bilaterally as a network formation game. Various equilibria arise. In most equilibria the financial system is resilient to the demise of some banks, but not the others. Equilibria in which there is no contagion can be supported as well. Moreover, I show that the welfare in equilibrium interbank networks is decreasing in the probability that contagion occurs. However, more bilateral agreements between banks do not necessarily improve welfare beyond the point when there is no contagion.

To study these issues, I build on the framework proposed by Allen and Gale (2000). In particular, I consider a three-period model, where the banking system consists of two identically sized regions. Banks raise deposits from consumers who are uncertain about their liquidity preferences, as in Diamond and Dybvig (1983). Each region is subject to liquidity shocks driven by consumers' liquidity needs. Liquidity shocks are negatively correlated across the two regions. In addition, there is a small probability that one of the banks, chosen at random, is affected by an early-withdrawal shock and liquidated prematurely.

Banks can perfectly insure against liquidity shocks by exchanging interbank deposits with banks in the other region. However, the connections created by swapping deposits expose the system to contagion when the early-withdrawal shock realizes. The loss that a bank induces when she is affected by the early-withdrawal shock is shared across her counterparties. This implies that as banks exchange more deposits, the loss on every deposit is smaller. The model predicts a connectivity threshold above which contagion does not occur. Reaching this connectivity threshold may require that banks swap deposits with other banks in the same region. I distinguish between a *liquidity network*, that smoothes out liquidity shocks in the banking system, and a *solvency network*, that provides insurance against contagion risk.

The distinction between liquidity links and solvency links is useful to study the incentives that banks have to insure against contagion. In particular, I study whether banks choose to form solvency links with other banks in the same region. When a bank has at least as many links as the connectivity threshold requires, then no contagion takes place if she is affected by the early-withdrawal shock. However, her counterparties still incur a loss on their deposits. When deciding to form solvency links, banks are willing to incur a small loss on their deposits, if they can avoid default. However, they are better off if contagion is averted without incurring any cost. This implies that banks have the incentive to free-ride on others' links. Because of this, many network structures can be stable in which full financial stability is not necessarily achieved.

In the main specification of the model, I show that at least half of the banks have swapped deposits with sufficiently many other banks in the same region. In other words, a systemic event in which all banks default if one bank is subject to an early-withdrawal shock occurs in at most half of the cases. Even then, I find that insuring against regional fluctuations in the fraction of early consumers through a liquidity network is optimal as long as the probability of the early-withdrawal shock is sufficiently small.

The equilibria in which all banks have sufficiently many links and no contagion occurs have the highest associated welfare. This is intuitive, as banks' assets are inefficiently liquidated if a systemic event occurs. However, in interbank networks in which contagion does not occur, welfare is not necessarily increasing with the number of links that banks have. This is because when each bank has more links, there are also more banks that incur losses on their deposits. However, since banks have already sufficiently many links, there is no benefit to offset this implicit cost.

Thus, increasing the connectivity of the interbank network is beneficial up to the point when there is no contagion in the financial system. The idea that an interconnected banking system may be optimal is supported by various other studies. Leitner (2005) discusses how the threat of contagion may be part of an optimal network design. His model predicts that it is optimal for some agents to bail out other agents, in order to prevent the collapse of the whole network. This form of insurance can also emerge endogenously, and I show that it is an equilibrium in a network formation game. Linkages between banks can also be efficient in the model of Kahn and Santos (2008) if there is sufficient liquidity in the financial system. Recently, Acemoglu et al. (2015) find that the types of financial networks that are most prone to contagious failures depend on the number of adverse shocks that affect the financial system.

The rationale for why a bank is willing to form solvency links with other banks in the same region and incur a loss on her deposits is that an early-withdrawal shock to any bank can have system-wide externalities. In particular, all banks default when a bank that has insufficient links is affected by an early-withdrawal shock. Banks are willing to pay a premium (i.e. incur a loss on their deposits) to avoid defaulting by contagion. Thus, a solvency interbank network can be interpreted as an alternative to formal insurance markets. Moreover, the network formation approach provides insights about the circumstances in which banks are willing to purchase protection, as well as about the premia they are willing to pay. From this perspective, the findings in this paper complement solutions relying on formal insurance arrangements previously proposed in the literature. For instance, Zawadowki (2013) shows, in the context of OTC traded contracts, that competitive insurance markets fail as banks find the premia for insuring against counterparty default too expensive. This is because banks do not internalize that the default of another bank which is not an immediate neighbor can nevertheless affect them in subsequent default waves. Insurance is unattractive in Kyiotaki and Moore (1997) as well, because of limited enforcement of contracts. Similarly, it has been shown that other formal arrangements, such as clearinghouses, may increase systemic risk either because they reduce netting efficiency (as in Duffie and Zhu, 2011) or they reduce dealers' incentives to monitor each other (as in Pirrong, 2009).

Starting with Allen and Gale (2000) there has been a growing interest in how different network structures respond to the breakdown of a single bank in order to identify which ones are more fragile: See, for instance, the theoretical investigation of Freixas et al. (2000) or Castiglionesi and Navarro (2007) , and the experimental study of Corbae and Duffy (2008). In parallel, the empirical literature has looked for evidence of contagious failures of financial institutions resulting from mutual claims they have on one another and has shown such interbank loans are unlikely to lead to sizable contagion in developed markets (Furfine, 2003; Upper and Worms, 2004). Other papers are concerned with whether interbank markets anticipate contagion. For instance, in Dasgupta (2004) contagion arises as an equilibrium outcome conditional on the arrival of negative interim information which leads to coordination problems among depositors and widespread runs, whereas Caballero and Simsek (2013) provide a model of market freezes when the complexity of a financial network increases the uncertainty about the health of trading counterparties and of their partners. More recently, Alvarez and Barlevy (2014) study mandatory disclosure of losses at financial institutions which are exposed to contagion via a network of interbank loans.

This paper is organized as follows. Section 2 introduces the model in its generality. Section 3 describes when contagion can occur and the payoffs that banks receive. Section 4 provides the equilibrium analysis. In section 5, I present an extension of the model to include small linking costs and discuss welfare implications. Section 6 concludes.

2 The Model

2.1 Consumers and liquidity preferences

The economy is divided into $2n$ sectors, each populated by a continuum of consumers. Consumers' preferences are described by a log-utility function. There are three time periods $t = 0, 1, 2$. Each agent is endowed with one unit of consumption good at date $t = 0$. Agents are uncertain about their liquidity preferences: they can be early consumers, who value consumption only at date 1, or they can be late consumers, who value consumption only at date 2.

The probability that an agent is a early consumer is q. I assume that the law of large numbers holds in the continuum, which implies that, on average, the fraction of agents

Probability	State/Sector	Region A 1	2	...	n	Region B $n+1$	$n+2$	...	$2n$
$(1-\varphi)/2$	S^1	p_H	p_H	...	p_H	p_L	p_L	...	p_L
$(1-\varphi)/2$	S^2	p_L	p_L	...	p_L	p_H	p_H	...	p_H
		$q+\chi$	q	...	q	q	q	...	q
φ	$\bar{S}$	q	$q+\chi$	...	q	q	q	...	q
		...	...	...	...	...	...	...	...
		q	q	...	q	q	q	...	$q+\chi$

Table 1: Distribution of shocks in the economy at date 1

that value consumption at date 1 is q. However, each sector experiences fluctuations of early withdrawals. With probability 1/2, in each sector there is either a high proportion, p_H, or a low proportion, p_L, of early consumers, so that $q = \frac{p_H + p_L}{2}$. In particular, the economy consists of two regions, $A = \{1, 2, ..., n\}$ and $B = \{n+1, n+2, ..., 2n\}$, such that fluctuations in the fraction of early consumers are perfectly correlated within each region and negatively correlated across regions. That is, when sectors in region A receive a high fraction, sectors in region B receive a low fraction, and the other way around.

Aggregate early-withdrawal shocks can affect the economy with a small, but positive, probability φ. In this case, the average fraction of early consumers is higher than q. For tractability, I assume that exactly one of the sectors receives a fraction $(q+\chi)$ of early consumers, whereas the others receive a fraction q. Each sector is equally likely to experience an early-withdrawal shock. The uncertainty in the liquidity preferences of consumers at date 1 is summarized in Table 1.[1]

At date 0 each of the sectors is ex-ante identical. All the uncertainty is resolved at date 1, when the state of the world is realized and commonly known. At date 2, the fraction of late consumers in each region will be $(1-p)$ where the value of p is known at date 1, as either p_H, p_L, q or $q+\chi$.

[1] Each realization of the aggregate early-withdawal shock in which sector k receives a fraction $(q+\chi)$ of early consumers represents a state $\bar{S}_k$ that occurs with probability $\frac{\varphi}{2n}$. I abuse notation and refer to the set of states $\{\bar{S}_k\}_k$ as state $\bar{S}$ that occurs with probability φ.

2.2 Banks, investment opportunities and interbank deposits

In each sector i there is a competitive representative bank. Agents deposit their endowment in their sector's bank. In exchange, they receive a deposit contract that promises a finite amount of consumption depending on the date they choose to withdraw their deposits, and that is, possibly, contingent on the state of the world. In particular, the deposit contract specifies that if they withdraw at date 1, they receive $C_{1i}^S \geq 1$, and if they withdraw at date 2, they receive $C_{2i}^S \geq C_{1i}^S$, where $S \in \{S^1, S^2, \bar{S}\}$.

Banks have two investment opportunities: a liquid asset with a return of 1 after one period, or an illiquid asset that pays a return of $r < 1$ after one period, or $R > 1$ after two periods. Let x_i and y_i be the per capita amounts that a bank i invests in the liquid and illiquid asset, respectively. In addition, banks can deposit funds at other banks in exchange for the same deposit contract offered to consumers. That is, for each unit that bank i deposits at bank j, she is promised C_{1j}^S if withdrawing at $t = 1$ and C_{2j}^S if withdrawing at time $t = 2$. Let z_{ij} denote the amount that bank i deposits at bank j, and z_i denote the total amount of interbank deposits that bank i holds.

Interbank deposits connect the banks in a *network* g. In particular, if bank i holds deposits at bank j, they are considered to have a *link* ij and to be *neighbors* in the network g. The set of neighbors of a bank i in the network g is $N_i(g) = \{j \in A \cup B \mid ij \in g \text{ for any } j \neq i\}$.

A set of contracts $\left(C_{1i}^S, C_{2i}^S\right)_{i \in A \cup B}$ and portfolio allocations $(x_i, y_i, z_i)_{i \in A \cup B}$ is *feasible* if it respects the feasibility constraints at date 0, 1, and 2, as follows.

At date 0, each bank i's portfolio must satisfy the following feasibility constraint

$$x_i + y_i + \sum_{j \in N_i(g)} z_{ij} = 1 + \sum_{\substack{j \\ i \in N_j(g)}} z_{ji}.$$

In other words, the amount that a bank i receives from depositors, 1, and other banks, $\left(\sum_{j,\, i \in N_j(g)} z_{ji}\right)$, can be invested in the liquid asset, the illiquid asset, or as deposits at other banks.

The feasibility constraint at date 1 requires that the payments to the early consumers and to banks that withdraw at date 1 equal the cash inflows from the liquid asset and the deposits withdrawn from other bank at date 1, in state S^1 and S^2. In state S^1, banks in region A withdraw deposits from banks in region A and B, whereas banks in region B withdraw deposits only from other banks in region B. Similarly, in state S^2, banks in region B withdraw deposits from banks in region A and B, whereas banks in region A withdraw deposits only from other banks in region A.

The feasibility constraint at date 2 requires that the payments to the late consumers and to banks that withdraw at date 2 equals the cash inflows from the illiquid asset and from the deposits withdrawn from other banks at date 2, in state S^1 and S^2. In state S^1, banks in region B withdraw deposits from banks in region A. Similarly, in state S^2, banks in region A withdraw deposits from banks in region B.

The feasibility constraints are, essentially, budget constraints. The condition that contracts and portfolio allocations respect the feasibility constraints in states S^1 and S^2, simply rules out that defaults occur in either of these states. However, I allow for the possibility that defaults occur in state $\bar{S}$, as banks are not required to have a balanced budget in this state.

2.3 Interbank networks

Various networks of interbank deposits can be considered. Figure 1 illustrates several patterns of connections between banks. In a *liquidity network* each bank has connections only with banks in the other region. In a *symmetric* network each bank has the same number of links. A symmetric liquidity network is shown in Figure 1(a), and a symmetric network is shown in Figure 1(b). Figure 1(c) represents a network in which each bank has the same number of connections with banks in the other region, but a different number of connections with banks in the same region. The links that banks have with other banks in the same region represent a *solvency network*. In a *complete network* each banks has connections with all other banks, as shown in Figure 1(d).

I will use the following notation throughout the paper. Let g_{ℓ,η_i} represent networks where each bank i has ℓ *liquidity links* with banks in the other region and η_i *solvency links* with banks in the same region. If all banks have same the number, η, of connections with banks in the same region, then the network is symmetric and denoted $g_{\ell,\eta}$. For instance $g_{n,n}$ represents the complete network, while $g_{n,0}$ represents a symmetric liquidity network in which each bank has links with all the bank in the other region, and no links with banks

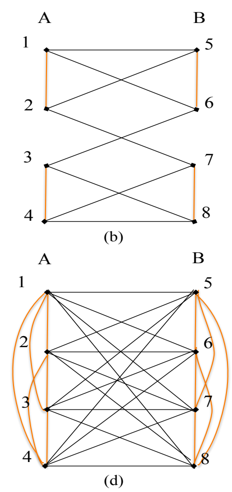

Figure 1: This figure illustrates various patterns of connections between banks. Panel (a) shows a symmetric liquidity network. Panel (b) illustrates a symmetric interbank network. Panel (c) illustrates an interbank network in which each bank has the same number of links with banks in the other region, but a different number of links with banks in the same region. Panel (d) illustrates a complete network.

in the same region.

In the analysis in the following section I focus on the case in which each bank has ℓ connections with banks in the other region. The main results, aside of Proposition 5, are derived for $\ell = n$.

3 Interbank Contagion

3.1 Optimal risk-sharing without aggregate uncertainty

The optimal risk sharing problem is well understood if there is no aggregate uncertainty about the average fraction of early consumers (i.e. when $\varphi = 0$). Allen and Gale (2000) characterize the optimal risk sharing allocation as the solution to a planning problem. They show that the optimal deposit contract (C_1^*, C_2^*) is state-independent, and maximizes the ex-ante expected utility of consumers. In the case when consumers have log-preferences it follows straightforwardly that

$$(C_1^*, C_2^*) = (1, R)\,. \tag{1}$$

Moreover, the optimal portfolio allocation requires that each bank i invests an amount x^* in the liquid asset in order to pay the early consumers and an amount $y^* = (1 - x^*)$ in the illiquid asset in order to pay the late consumers

$$(x^*, y^*) = (qC_1^*, (1-q)C_2^*/R). \tag{2}$$

As Allen and Gale (2000) show, an interbank system can decentralize the planner's solution, if each bank deposits with banks in the other region a total amount $z^* = p_H - q = q - p_L$, at date 0.[2]

The planner's solution can be implemented by any symmetric liquidity network, $g_{\ell,0}$, where each bank has liquidity links with $\ell \in \{1, 2, ..., n\}$ banks in the other region, and the amount of deposits exchanged between any two banks is $\frac{z^*}{\ell}$ at date 0. Moreover, any network g_{ℓ,η_i} in which any two banks that have a link exchange $\frac{z^*}{\ell}$ as deposits at date 0 can also decentralize the planner solution. This is because deposits exchanged with banks in the same region mutually cancel out in either state S^1 or S^2, as they do not provide insurance against regional liquidity fluctuations. At the same time, although a bank can deposit more than z^* with banks in the other region, in an interbank network g_{ℓ,η_i} insurance against liquidity shocks can be achieved only if banks that have a liquidity shortage withdraw a net amount of $\frac{z^*}{\ell}$ from the banks that have a liquidity surplus.

It is straightforward to see that the contract (1) and the portfolio (2) respect the feasibility constraints introduced in Section 2.2. Moreover, no defaults occur when $\varphi = 0$.

3.2 Defaults and the contagion mechanism

Next, I consider the case of $\varphi > 0$, when each bank incurs an early-withdrawal shock with probability $1/2n$ in state $\bar{S}$. I show that defaults and contagion can occur in a given interbank network, g_{ℓ,η_i}, in which banks offer the deposit contract (1), hold the portfolio (2), and insure an amount z^*/ℓ as bilateral deposits with other banks. I assume that each bank exchanges z^*/ℓ deposits with both banks in the other region, as well as banks in the

[2] Exchanging interbank deposits *ex-ante* in order to insure against liquidity shocks may be seen as uncoventional. Acharya et al. (2012) emphasize some of the problems that occurs when liquidity transfers occur *ex-post*. For instance, surplus banks may strategically under-provide lending to induce inefficient sales of assets from needy banks.

same region. Although it is feasible to consider that banks in the same region exchange a different amount as deposits, this assumption simplifies our analysis without losing any insights.

A bank that needs to repay $(q+\chi)\,C_1^*$ to the early consumers in state $\bar{S}$ does not have sufficient liquidity at date 1, as the proceeds from the liquid asset are $x^* = qC_1^*$. Hence, the bank must liquidate either some of its interbank deposits and/or the illiquid asset. As in Allen and Gale (2000), I assume that the costliest in terms of early liquidation is the illiquid asset, followed by interbank deposits:

$$\frac{C_2^*}{C_1^*} < \frac{R}{r}. \tag{3}$$

This implies that the bank liquidates deposits in other regions before it liquidates the illiquid asset. In state $\bar{S}$, liquidating interbank deposits, although beneficial, as I describe below, does not generate liquidity. Thus, the bank must liquidate at least part of the illiquid asset in order to meet withdrawals from early consumers.

A bank that liquidates the illiquid asset prematurely, affects negatively the consumption of late withdrawers. In fact, if too much of the illiquid asset is liquidated early, the consumption of late consumers may be reduced to a level below C_1^*. In this case, the late consumers gain more by imitating the early consumers and withdrawing their investment from the bank at date 1. This induces a run on the bank. The maximum amount of illiquid asset that can be liquidated without causing a run is given by

$$b \equiv y^* - \frac{(1-q)C_1^*}{R}, \tag{4}$$

or, substituting from (2),

$$b = (1-q)\frac{(C_2^* - C_1^*)}{R}.$$

For the remainder of the paper, I assume that

$$\chi > \frac{r \cdot b}{C_1^*}. \tag{5}$$

In other words, the amount that can be obtained at date 1 by liquidating the long asset without causing a run, $r \cdot b$, is not sufficient to repay the additional fraction, χ, of early depositors. Thus, a bank which offers the deposit contract (1), and holds the portfolio (2) cannot repay C_1^* to depositors that withdraw at date 1, if she incurs an early-withdrawal shock in state $\bar{S}$. In this case, the bank defaults, and its portfolio of assets is liquidated at the current value and distributed equally among creditors.

Suppose that bank k is affected by the early-withdrawal shock in state $\bar{S}$, when the network of interbank deposits is g_{ℓ,η_i}. The three assets in the bank k's portfolio yield different returns upon liquidation in period 1. First, the liquid asset pays a return of 1. Second, the illiquid asset, pays a return of $r < 1$ if liquidated early. And lastly, the interbank deposits held at a bank j yield a return, $C_{1j}^d \leq C_1^*$. On the liability side, a bank has to pay its depositors, normalized to 1 and at the same time to repay its interbank creditors that add up to $(\ell + \eta_k) \cdot \frac{z^*}{\ell}$. This yields at date 1 a new return per unit of good deposited in bank k equal to

$$C_{1k}^d(g_{\ell,\eta_i}) = \frac{x^* + ry^* + \sum_{j \in N_k(g)} \frac{z^*}{\ell} C_{1j}^d(g)}{1 + (\ell + \eta_k) \cdot \frac{z^*}{\ell}}. \tag{6}$$

The return that bank k pays on early withdrawals, C_{1k}^d, depends on the network g_{ℓ,η_i} of interbank deposits, as it is described in Section 3.3. However, when unnecessary, I suppress the dependency in the notation and take as implicit that $C_{1k}^d(g_{\ell,\eta_i}) = C_{1k}^d$ for any k.

Following the default of bank k, subsequent defaults are possible. In particular, if $C_{1k}^d < C_1^*$, then a bank j that has deposits at bank k incurs a loss of value on its deposits, or a *loss given default* (henceforth, LGD). The LGD that bank j incurs in a network g_{ℓ,η_i} when bank k has been liquidated is given by

$$LGD_{jk}(g_{\ell,\eta_i}) = \frac{z^*}{\ell}(C_1^* - C_{1k}^d), \tag{7}$$

or, substituting (2) in (6),

$$LGD_{jk}(g_{\ell,\eta_i}) = \frac{z^*}{\ell} \left(\frac{(1-q)\left(C_1^* - \frac{r}{R} C_2^*\right)}{1 + (\ell + \eta_k) \cdot \frac{z^*}{\ell}} + \frac{\sum_{l \in N_k(g_{\ell,\eta_i})} \frac{z^*}{\ell} \left(C_1^* - C_{1l}^d\right)}{1 + (\ell + \eta_k) \cdot \frac{z^*}{\ell}} \right). \tag{8}$$

A positive LGD triggers the early liquidation of the illiquid asset to meet early withdrawals. If bank j needs to liquidate an amount of the illiquid asset higher than b, she fails, as explained above. When a bank fails by contagion, its portfolio of assets is also liquidated at the current value and distributed equally among creditors.[3] In contrast, if

[3] This explains why bank k, that has initially incured the early-withdrawal shock, may reiceve a return on its interbank deposits lower than C_1^*, as reflected in (6).

the amount of the illiquid asset liquidated at date 1 is below the threshold b, bank j does not default and returns C_1^* for early withdrawals. Nevertheless, it will be costly for the late consumers, as their consumption is now reduced to $C_{2j}^d < C_2^*$. Thus, b given by (4) represents a *contagion threshold*, as it is the maximum amount of illiquid asset that a bank can liquidate without causing a run.

Two distinct implications follow from (8). First, the loss given default LGD_{jk} is increasing in the amount of deposits, $\frac{z^*}{\ell}$, exchanged between the two banks. In other words, the more liquidity links, ℓ, each bank has with banks in the other region, the smaller is the loss given default. Second, the more links bank k has with banks that are able to repay C_1^* for deposits, the smaller is the loss LGD_{jk} it induces to a neighbor j. This effect is independent of the amount of deposits exchanged between banks, and it arises, for instance, when keeping the number of links, ℓ, with banks in the other region constant and increasing the number of solvency links, η_k, with banks in the same region. There are two channels that explain this second implication. Everything else equal, increasing the number of links with banks that are able to repay a return for deposits of C_1^* at date 1, mechanically decreases the loss that bank k induces to its neighbors. This is because, the denominator in (8) increases, while the numerator remains the same. In other words, the loss that bank k induces when it fails is redistributed across more counterparties. At the same time, a positive spiral that further reduces the loss-given-default may arise. For instance, the LGD_{jk} incurred by neighbor j may decrease sufficiently, such that bank j does not default by contagion and is also able to repay bank k a return C_1^* per-unit of deposits. Then, the loss that k induces to its neighbors decreases even further, reaching its minimum when none of the neighbor banks defaults

$$LGD_{jk}^{\min}(\eta_k) = \frac{z^*}{\ell} \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1 + (\ell + \eta_k)\frac{z^*}{\ell}}, \tag{9}$$

for any $j \in N_k(g_{\ell,\eta_i})$.

Thus, the contract (1) and portfolio (2) involve a trade-off. Because defaults involve the liquidation of the illiquid asset, there is a utility loss in state $\bar{S}$. A different arrangement, in which banks invest more in the liquid asset and insure less through the interbank system may be desirable. A bank that has cash reserves, can avoid a run if it incurs an early-withdrawal shock. At the same time, she holds fewer deposits with other banks in the system and incurs a lower loss-given-default. This way, banks can avoid default in state $\bar{S}$, at the expense of providing a lower utility to consumers in states S^1 and S^2.

In the remainder of the section I discuss the payoffs that banks expect to receive in each state of the world[4], under the assumption that each bank offers the deposit contract (1) and holds the portfolio (2). In section 4, I come back to this issue and show that banks find it optimal to offer the deposit contract (1) and to hold the portfolio (2) provided φ is sufficiently small. For now, I start by describing the payoffs in state $\bar{S}$.

3.3 Expected payoffs

Consider as before that in state $\bar{S}$ bank k is affected by the early-withdrawal shock, and suppose that the number of connections that bank k has with banks is in the same region, η_k, satisfies the following inequality

$$\frac{z^*}{\ell} \frac{(1-q)\left(C_1^* - \frac{r}{R} C_2^*\right)}{1 + (\ell + \eta_k)\frac{z^*}{\ell}} \leq r \cdot b. \tag{10}$$

The left hand side of the inequality represents the loss-given default that a neighbor of bank k receives in a network g_{ℓ,η_i}, provided all k's neighbor banks repay a return C_1^* for the deposits they have received from bank k. At the same time, repaying C_1^* is indeed consistent with the loss-given-default that a neighbor j of bank k receives in the network g_{ℓ,η_i}, as the maximum amount of the illiquid asset that can be liquidated without causing a run, b, is larger than $\frac{LGD_{jk}^{\min}(g_{\ell,\eta_i})}{r}$. This implies that in the network g_{ℓ,η_i} each bank returns for early withdrawals C_1^*, except for bank k which returns

$$C_{1k}^d(\eta_k) = C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R} C_2^*\right)}{1 + (\ell + \eta_k)\frac{z^*}{\ell}}. \tag{11}$$

Moreover, each of the $[2n - (\ell + \eta_k + 1)]$ banks that do not have a connection with bank k returns C_2^* for late withdrawals. However, each of the $(\ell + \eta_k)$ banks that have a connection with bank k must liquidate an amount of $\frac{LGD_{jk}^{\min}(g_{\ell,\eta_i})}{r}$ from the illiquid asset and returns for late withdrawals

$$C_{2j}^d(\eta_k) = C_2^* - \frac{z^*}{\ell} \frac{\frac{R}{r} C_1^* - C_2^*}{1 + (\ell + \eta_k)\frac{z^*}{\ell}}, \tag{12}$$

[4] Banks are perfectly competitive and make zero-profits. However, because banks maximize the expected utility of consumers, I abuse terminology and use banks' payoffs to refer to banks' consumers payoffs.

for any $j \in N_k(g_{\ell,\eta_i})$, with $C^d_{2j}(\eta_k) \geq C^*_1$.

In contrast, if inequality (10) does not hold, then any neighbor of bank k defaults by contagion. This is because even if all $(\ell + \eta_k)$ neighbors of bank k repay C^*_1, they still need to liquidate too much of the illiquid asset. Clearly, this implies that it is impossible for any of them to repay C^*_1 to start with, and the realized loss-given-default is even higher.

When inequality (10) does not hold, the payoffs that banks receive depend on the entire network structure. The procedure to finding the solution involves a sequence of steps. First, find the return that bank k and each of its neighbors $j \in N_k(g_{\ell,\eta_i})$ repay for early withdrawals. For this, solve the system of $(\ell + \eta_k + 1)$ equations implied by (6), under the assumption that the remaining banks that are not neighbors of k do not default and are able to repay C^*_1. Second, verify that these banks are indeed able to repay C^*_1, given the losses-given-default implied at the first step. If all banks that are not neighbors of k are able to repay C^*_1, then the solution is the one found at the first step. Otherwise, if a subset of m of these banks are not able to repay C^*_1, solve the system of $(\ell + \eta_k + 1 + m)$ equations implied by (6), under the assumption that the remaining banks do not default and are able to repay C^*_1. Verify that this is indeed consistent with the solution found. Otherwise, continue the procedure until all banks default. This solutions concept is similar to the algorithm that Elliott, Golub and Jackson (2013) propose to characterize waves of defaults in a network of liabilities (which is a generalization of the algorithm in Eisenberg and Noe, 2001).

Although a solution for a general network g_{ℓ,η_i} is difficult to characterize, the following proposition describes the payoffs that banks receive when $\ell = n$, for any number of links η_i that a bank i has with banks in the same region. Moreover, for the remainder of the paper I assume as well that $\ell = n$, and relax this assumption when I discuss incomplete liquidity networks in Section 5.

Proposition 1 *Consider any interbank network g_{n,η_i} in which each bank offers the deposit contract (1) and holds the portfolio (2). Let $\bar{\eta}$ be the smallest positive integer that satisfies the inequality*

$$\frac{z^*}{n}\frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+\bar{\eta})\frac{z^*}{n}}\leq r\cdot b. \tag{13}$$

If the bank that is subject to the early-withdrawal shock in state $\bar{S}$ has less than $\bar{\eta}$ connections, then each bank returns per unit of deposit at date 1

$$C_1^d=C_1^*-(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right). \tag{14}$$

The proof for Proposition 1 follows in two steps. First, I show that if the bank that is subject to the early-withdrawal shock has less than $\bar{\eta}$ connections, then all the other $(2n-1)$ banks fail by contagion. Importantly, this is independent of how many connections each bank has with banks in the same region, as the result holds for any network g_{n,η_i}. Second, I find the vector of returns $\left(C_{1i}^d\right)_{i\in A\cup B}$ that is a fixed point of the system of $2n$ equations implied by (6).

Proposition 1 easily generalizes to any network g_{ℓ,η_i} in which all banks default in state $\bar{S}$. In particular, let $\bar{\eta}(\ell)$ be the smallest integer for which inequality (10) holds. Consider parameters such that if the bank subject to the early-withdrawal shock has less than $\bar{\eta}(\ell)$ connections, then all the other $(2n-1)$ banks fail by contagion. Then all banks return C_1^d per unit of deposit as given by (14).

At this stage I can characterize the payoffs that each bank i expects to receive in a network g_{n,η_i}. In both states S^1 and S^2, each bank has with probability half either a high fraction, p_H, of early consumers or a low fraction, p_L, of early consumers. Hence, consumers expect to receive in each of these states

$$qu\left(C_1^*\right)+(1-q)\,u\left(C_2^*\right).$$

In state $\bar{S}$, consumers' expected utility depends on how many banks have at least $\bar{\eta}$ connections with banks in the same region and on whether bank i, itself has at least $\bar{\eta}$ connections with banks in the same region. Let $H(g_{n,\eta_i})=\left\{j\in A\cup B|\eta_j\geq\bar{\eta}\right\}$ be the set of banks that have at least $\bar{\eta}$ connections, and let $h=\left|H(g_{n,\eta_i})\right|$ be the number of banks that have at least at least $\bar{\eta}$ connections. This implies that there are $(2n-h)$ banks that each induces the default of the entire system, when affected by the early-withdrawal shock. Moreover, let $H_i(g_{n,\eta_i}) = \{j \in N_i(g_{n,\eta_i}) | \eta_j \geq \bar{\eta}\}$ be the set of neighbors of bank i that have at least $\bar{\eta}$ connections. This implies that whenever a bank $j \in H_i(g_{n,\eta_i})$ is affected by the early-withdrawal shock, bank i returns to its late consumers $C_{2i}^d(\eta_j)$ as given by (12) for $\ell = n$. In addition, the return that bank i pays in case it is affected by the early-withdrawal shock depends on how many connections it has. Thus, if $i \in H(g_{n,\eta_i})$, then it returns $C_{1i}^d(\eta_i)$ as given by (11) for $\ell = n$. Otherwise, it returns C_1^d as given by (14).

The expected payoff of a given bank i is as follows

$$\begin{aligned}\pi_i(g_{n,\eta_i}) &= (1-\varphi)\left[q\ln(C_1^*) + (1-q)\ln(C_2^*)\right] \\ &\quad +\varphi\Bigg[\frac{2n-h}{2n}\ln\left(C_1^d\right) + \frac{1}{2n}\sum\nolimits_{j\in H_i(g_{n,\eta_i})}\left(q\ln(C_1^*) + (1-q)\ln\left(C_{2i}^d(\eta_j)\right)\right) \\ &\quad\quad +\frac{1}{2n}\sum\nolimits_{j\in H(g_{n,\eta_i})\backslash H_i(g_{n,\eta_i})}\left(q\ln(C_1^*) + (1-q)\ln(C_2^*)\right)\Bigg],\end{aligned} \tag{15}$$

if $i \notin H(g_{n,\eta_i})$, and

$$\begin{aligned}\pi_i(g_{n,\eta_i}) &= (1-\varphi)\left[q\ln(C_1^*) + (1-q)\ln(C_2^*)\right] \\ &\quad +\varphi\Bigg[\frac{2n-h}{2n}\ln\left(C_1^d\right) + \frac{1}{2n}\sum\nolimits_{j\in H_i(g_{n,\eta_i})}\left(q\ln(C_1^*) + (1-q)\ln\left(C_{2i}^d(\eta_j)\right)\right) \\ &\quad\quad +\frac{1}{2n}\sum\nolimits_{j\in H(g_{n,\eta_i})\backslash H_i(g_{n,\eta_i})}\left(q\ln(C_1^*) + (1-q)\ln(C_2^*)\right) \\ &\quad\quad +\frac{1}{2n}\ln\left(C_{1i}^d(\eta_i)\right)\Bigg],\end{aligned} \tag{16}$$

if $i \in H(g_{n,\eta_i})$.

4 Endogenous Solvency Networks

When there is a risk of contagion in the financial system, banks can take actions to insure against it. The decisions that each bank i must consider at date 0 in order to maximize the expected utility of consumers, consist of a deposit contract (C_{1i}^S, C_{2i}^S) for early and late withdrawals, a portfolio of liquid and illiquid assets and interbank deposits (x_i, y_i, z_i^*), a set of liquidity links with banks in the other region, and a set of solvency links with banks in the same region. In particular, consider the following timing of events at date 0. At stage 1, each bank chooses a deposit contract and a portfolio allocation. At stage 2, each bank chooses a set of links with banks in the other region, specifying for each link the amount of interbank deposits she wants to insure with the respective counterparty. At stage 3, each bank chooses a set of links with banks in the same region, specifying as well, for each link, the amount of interbank deposits it wants to insure with the respective counterparty. At each stage, each bank takes the decisions at previous stage(s), as well, the decisions of other at the current stage as given. Furthermore, at each stage banks understand the consequences of their current decisions on the choices to be made at future stage(s). In making choices, at each stage banks could reason backwards and choose the deposit the contract that maximizes the expected utility of consumers.

The difficulty with this approach is that decisions at stage 2 (and 3) involve both a set of links, as well as an amount of deposits for each link. This prevents agents from taking decisions unilaterally, or even bilaterally. For instance, consider a symmetric liquidity network $g_{\ell,0}$ where each bank has ℓ links with banks in the other region, and any two banks that have a link exchange an amount $\frac{z^*}{\ell}$ as deposits. Suppose that at stage 1, banks offer the deposit contract (C_1^*, C_2^*), hold the portfolio (x^*, y^*) and insure an amount $z^* = (p_H - q)$ as interbank deposits. Then, a bank that is considering decreasing the number of links with banks in the other region from ℓ to $\ell - 1$, must re-adjust the amount of deposits it exchanges with at least one other bank (from $\frac{z^*}{\ell}$ to $\frac{2z^*}{\ell}$), in order to respect the feasibility constraints described in Section 2.2. However, the re-adjustment cannot be done unilaterally as her counterparty has an excess of incoming deposits, unless she re-adjusts the deposits she exchanges with other counterparties in order to respect her feasibility constraints.

The approach I take instead is similar to the solution method in Dasgupta (2004). First, I analyze the equilibrium networks on the continuation path induced by the deposit contract (1) and the portfolio (2), when each bank has n connections with banks in the other region and any two banks that have a link exchange $\frac{z^*}{n}$ in deposits. Second, I show that each bank's best response is to offer the deposit contract (1) and the portfolio (2) when all other banks offer the deposit contract (1) and the portfolio (2), at least when the probability of state $\bar{S}$ is sufficiently small.[5]

[5] A complimetary approach is to consider that banks first choose a set of links and then choose a deposit contract and a portfolio allocation. Acemoglu et al. (2015) explore this route using a solution strategy similar to mine. In particular, they solve for the equilibrium interbank lending decisions and interest rates, given that banks are connected in an exogenous network which restricts the amount they can borrow from

To analyze the equilibrium networks, I model the interaction between banks in the same region as a game in which banks choose with whom to form links. Such a game is called a network formation game. The formation of a link requires the consent of both parties involved, since banks mutually exchange deposits. However, the severance of a link can be done unilaterally, as deposits can be withdrawn on demand by either of the banks involved in a link (in this case, deposits will be restituted to both banks, although only one party exercises the claim). To identify equilibrium networks, I use the concept of pairwise stability introduced by Jackson and Wolinsky (1996).[6]

Definition 1 *An interbank network g is pairwise stable if*

(i) for any pair of banks i and j that are linked in the interbank network g, neither of them has an incentive to unilaterally sever their link ij. That is, the expected profit each of them receives from deviating to the interbank network $(g - ij)$ is not larger than the expected profit that each of them obtains in the interbank network g ($\pi_i(g - ij) \leq \pi_i(g)$ and $\pi_j(g - ij) \leq \pi_j(g)$);

(ii) for any two banks i and j that are not linked in the interbank network g, at least one of them has no incentive to form the link ij. That is, the expected profit that at least one of them receives from deviating to the interbank network $(g + ij)$ is not larger than the expected profit that it obtains in the interbank network g (if $\pi_i(g + ij) > \pi_i(g)$ then $\pi_j(g + ij) < \pi_j(g)$).

The result below describes whether forming or severing links is profitable for any given pair of banks i and j, depending on the number of links that they have in a network g_{n,η_i}. The derivations are shown in Appendix B.

Proposition 2 *The marginal payoffs of forming and severing a link, respectively, for bank i in a network g_{n,η_i} are characterized by the following properties:*

each other. They study then whether equilibrium outcomes are optimal or not, depending on the network structure.

[6] This concept has been applied by Zawadowski (2013) to analyze the stability of bilateral insurance contracts in the context of OTC markets. More recently, Farboodi (2014) uses an extension of the concept that allows for group deviations to explain intermediation in interbank markets.

Property 1: $\pi_i(g_{n,\eta_i} + ij) < \pi_i(g_{n,\eta_i})$ *if* $\eta_i \geq \bar{\eta}$ *or* $\eta_i \leq \bar{\eta} - 2$ *for any* $\eta_j \geq \bar{\eta}$.

Property 2: $\pi_i(g_{n,\eta_i} - ij) > \pi_i(g_{n,\eta_i})$ *if* $\eta_i \geq \bar{\eta} + 1$ *or* $\eta_i \leq \bar{\eta} - 1$ *for any* $\eta_j \geq \bar{\eta} + 1$.

Property 3: $\pi_i(g_{n,\eta_i} + ij) > \pi_i(g_{n,\eta_i})$ *if* $\eta_i \geq \bar{\eta} - 1$ *for any* $\eta_j \leq \bar{\eta} - 1$, *or for any* η_i *if* $\eta_j = \bar{\eta} - 1$.

Property 4: $\pi_i(g_{n,\eta_i} - ij) < \pi_i(g_{n,\eta_i})$ *if* $\eta_i \geq \bar{\eta}$ *for any* $\eta_j \leq \bar{\eta}$, *or for any* η_i *if* $\eta_j = \bar{\eta}$.

Property 5: $\pi_i(g_{n,\eta_i} + ij) < \pi_i(g_{n,\eta_i})$, *if* $\eta_i = \bar{\eta} - 1$ *and* $\eta_j \geq \bar{\eta}$, *for any* $r < \frac{z^*(n+\bar{\eta})}{n((2z^*+1)(n+\bar{\eta})-1)}$.

Property 6: $\pi_i(g_{n,\eta_i} - ij) > \pi_i(g_{n,\eta_i})$, *if* $\eta_i = \bar{\eta}$ *and* $\eta_j \geq \bar{\eta} + 1$, *for any* $r < \frac{z^*(n+\bar{\eta})}{n((2z^*+1)(n+\bar{\eta})-1)}$.

Property 7: $\pi_i(g_{n,\eta_i} + ij) = \pi_i(g_{n,\eta_i})$ *if* $\eta_i \leq \bar{\eta} - 2$ *and* $\eta_j \leq \bar{\eta} - 2$.

Property 8: $\pi_i(g_{n,\eta_i} - ij) = \pi_i(g_{n,\eta_i})$ *if* $\eta_i \leq \bar{\eta} - 1$ *and* $\eta_j \leq \bar{\eta} - 1$.

The main trade-offs that banks take into account when forming or severing links are as follows. On the one hand, there is an implicit cost of having a link with a bank that has at least $(\bar{\eta} + 1)$ links (or forming a link with a bank that has $\bar{\eta}$ links). If she incurs an early-withdrawals shock, her neighbors, though they do not default, are not able to return C_2^* to the late consumers. This effect is captured by Properties $1-2$. On the other hand, banks are willing to sacrifice some utility from the late consumers if they can avoid defaulting by contagion. Thus, banks find it beneficial to maintain a link with a bank that has $\bar{\eta}$ links (or form a link with a bank that has $(\bar{\eta} - 1)$ links). This effect is captured by Properties $3 - 4$.

The trade-off between the cost and benefit of linking is more complex in two cases, which are characterized by Properties $5 - 6$. When bank i has $\eta_i = \bar{\eta} - 1$ links, forming a new a link with a bank j that has $\eta_j \geq \bar{\eta}$ links implies that bank i is able to repay $C_{1i}^d(\eta_i + 1)$, as opposed to C_1^d, if she is affected by the early-withdrawal shock. However, bank i can return only $C_{2i}^d(\eta_j + 1)$ for late withdrawals, as opposed to C_2^*, if bank j is affected by the early-withdrawal shock. Similarly, when bank i has $\eta_i = \bar{\eta}$ links, severing a a link with a bank j that has $\eta_j \geq \bar{\eta} + 1$ links implies that bank i returns only C_1^d, as opposed to $C_{1i}^d(\eta_i)$, if she is affected by the early-withdrawal shock. However, bank i is able to return C_2^* for late withdrawals, as opposed to $C_{2i}^d(\eta_j)$, if bank j is affected by the early-withdrawal shock. When r is sufficiently small, the utility gained when bank i is affected by the early-withdrawal shock is not sufficient to compensate for the utility lost when bank j is affected by the liquidity shock. As r increases, whether bank i finds it profitable to form or severe a link with bank j that has at least $\bar{\eta}$ links is ambiguous, and it depends on exactly how many links bank j has. In particular, the more links bank j has, the more attractive is for bank i to form or maintain a link with bank j. In any case, this does not hinder the characterization of stable networks in terms of how many banks have at least $\bar{\eta}$ connections.

Lastly, there is no utility gain nor loss from forming a new link when banks have less than $(\bar{\eta} - 2)$ links, or severing an existing link when banks have less than $(\bar{\eta} - 1)$ links. This effect is captured by Properties $7 - 8$.

Typically, in many models of network formation and otherwise, multiple equilibria arise when agents are indifferent between taking and not taking an action. The number of equilibria is particularly high in models of network formation simply because of the dimensionality of the problem. Not surprisingly, the current model features multiple equilibria as well. For instance, given Properties $7 - 8$ described above, any network g_{n,η_i} in which a bank i has $\eta_i \leq \bar{\eta} - 2$ is pairwise stable. Clearly, there is a concern whether these equilibria are robust, since a bank may be tempted to switch to a payoff-equivalent link profile.

Generally, eliminating cases where players are indifferent results in a significant reduction of equilibria. Several solutions have been explored in the literature. For instance, Bala and Goyal (2000) impose a stronger equilibrium concept, which requires that each agent gets a strictly higher payoff from his current linking strategy than he would with any other strategy, whereas Hojman and Szeidl (2008) impose conditions on the payoff function to reduce the set of equilibrium networks. A useful benchmark that minimizes free-riding for the current model is to assume that if two banks are indifferent between forming and not forming the link, then they form the link. Similarly, if two banks are indifferent between keeping and severing a link, then they keep the link. In section 5.1, I discuss the effects of imposing a small cost for linking, which would imply that if two banks are indifferent between forming and not forming the link, then they do not form the link, whereas if two banks are indifferent between keeping and severing a link, then they severe the link.

Properties 1 to 8 characterize the benefits and the costs of forming or severing links for any given pair of banks i and j, spanning all combinations of the number of links that they have in a network g_{n,η_i}. For instance, consider the case of bank i, with $\eta_i = \bar{\eta} + 1$, and j, with $\eta_j = \bar{\eta} - 2$, that do not have a link. Property 3 characterizes the incentives for bank i to form the link, while Property 1 characterizes the incentives for bank j to form the link. These properties are useful in deriving the following result.

Proposition 3 *Let g_{n,η_i} be a pairwise stable network. Then there are at least* $\max\{2(n-\bar{\eta}), n\}$ *banks that have $\bar{\eta}$ connections with banks in the same region.*

The main intuition for this result is as follows. The threshold number of connections $\bar{\eta}$ determines what externality each bank has on the entire banking system. When a bank that is affected by the early-withdrawal shock has less than $\bar{\eta}$ connections with banks in the same region, all banks default by contagion as Proposition 1 shows. However, if a bank that is affected by the early-withdrawal shock has at least $\bar{\eta}$ connections with banks in the same region, then only the consumption of the late consumers of her neighbors is negatively affected. These two types of externalities drive banks' incentives to form or severe links. In particular, banks weigh the benefit of forming links that allow them to avoid defaulting by contagion, against the implicit cost that late consumers incur. Properties $3-4$ imply that banks are willing to incur the implicit cost for the late consumers, if they can avoid default. However, they are better off if they free-ride on others' links.

The trade-off that a link involves is best illustrated by the following case. Consider two banks, i with $\eta_i = \bar{\eta} - 1$, and j with $\eta_j \geq \bar{\eta}$, that do not have a link. It follows from Property 3 that bank j benefits from forming with i, as she exchanges a situation when the failure of i induces its own failure, for a situation when the failure of i results in a lower utility for her late consumers. However, bank i does not internalize the effects that its own failure on other banks. That is, the utility gained when bank i is affected by the early-withdrawal shock is not sufficient to compensate for the utility lost when bank j is affected by the liquidity shock, if r is not too large. In consequence, the link between i and j is not formed, as the formation of a link requires the consent of both banks.

However, the result described in Proposition 3 does not depend on r being small enough. This is because the trade-off described above generalizes to most pairs of banks. In particular, Properties $3-4$ and $7-8$ imply that only banks that have less than $\bar{\eta}-1$ links unequivocally benefit from forming or maintaining links with each other. Thus, a bank that has less than $\bar{\eta}-1$ links seeks to form links with other banks, until she has at least $\bar{\eta}$ links and no other banks accepts a link with her, or until the only banks with whom she does not have a link have at least $\bar{\eta}$ links. In consequence, there is always a set of banks that have at least $\bar{\eta}$ links.

Proposition 3 has significant implications for the stability of the banking system. In particular, when $\bar{\eta}$ is small, in equilibrium most of the banks have sufficient links to prevent a shock in one of the institutions spreading through contagion. From (13) it follows that the higher r or R is, the smaller is $\bar{\eta}$. However, even as $\bar{\eta}$ increases, in a stable network, at least half the banks will have a sufficiently many links such that the losses they may generate are small enough.

The following result characterizes the probability of systemic risk in an equilibrium interbank network.

Corollary 1 *(i) Let g_{n,η_i} be a pairwise stable network. Then the probability that all banks default is at most* $\min\{\frac{\varphi\bar{\eta}}{n}, \frac{\varphi}{2}\}$.

(ii) There exist pairwise stable networks in which all banks have $\bar{\eta}$ links and the probability that all banks default is 0.

This result identifies an upper bound on the probability that all banks default by contagion in state $\bar{S}$. Corollary 1 is an immediate consequence of Proposition 3, and a proof is omitted. In a network g_{n,η_i} all banks default by contagion when a bank that has less $\bar{\eta}$ links with banks in the same region is affected by the early-withdrawal shock. The result follows since there are at most $\min\{2\bar{\eta}, n\}$, and each banks is affected by an early-withdrawal shock with probability $1/n$.

Proposition 3 identifies many interbank networks that can be pairwise stable. The following result refines the characterization, and, more importantly, provides a ranking of equilibria based on their implied expected welfare.

Proposition 4 *Let* $r < \frac{z^*(n+\bar{\eta})}{n((2z^*+1)(n+\bar{\eta})-1)}$. *Then, in any pairwise stable network each bank has at most* $\bar{\eta}$ *links. Moreover, the expected welfare in pairwise stable networks is increasing in the number of banks that have* $\bar{\eta}$ *links.*

While contagion can occur in any pairwise stable interbank network in which at least one bank has less than $\bar{\eta}$ links, the probability that all banks default decreases with the number of banks that have $\bar{\eta}$ links. The result follows as the welfare loss induced when a bank that has less than $\bar{\eta}$ links is affected by the early-withdrawal shock is smaller then the welfare loss when a bank that has $\bar{\eta}$ links is affected by the early-withdrawal shock. Moreover, an interbank network in which all banks have $\bar{\eta}$ links and contagion never arises can be supported as an equilibrium.[7] Such interbank network insures the highest welfare of all pairwise stable networks.

Up to now, I have analyzed equilibrium networks on the continuation path induced by the deposit contract (1) and the portfolio (2). Clearly it is important to insure that banks find it optimal to offer the deposit contract (1), and to hold the portfolio (2). The following proposition shows that this is the case, at least if the probability of state $\bar{S}$ is sufficiently small.

Proposition 5 *There exists* $\bar{\varphi} \in (0, 1)$ *such that there is an equilibrium in which each bank offers the deposit contract (1) and holds the portfolio (2), for any* $\varphi \leq \bar{\varphi}$.

Rather than deriving the equilibrium deposit contract and portfolio allocation that maximizes the expected utility of consumers for each φ, Proposition 5 shows that each bank's best response is to offer the deposit contract (1) and the portfolio (2) when all other banks offer the deposit contract (1) and the portfolio (2) if φ is sufficiently small.

The intuition is as follows. Because the deposit contract (1) and the portfolio (2) is optimal when $\varphi = 0$, a bank that invests more in the liquid asset provides a lower utility to her consumers in state S^1 and S^2. In Lemma 1 in the appendix, I show that a bank that deviates from the contract (1) and the portfolio (2), while all other banks continue to choose the contract (1) and the portfolio (2), must be self-sufficient. That is, the bank

[7] The existence of such an equilibrium network is conditional on whether symmetric networks in which each banks has a total of $n + \bar{\eta}$ links exist. Lovasz (1979) discusses in detail conditions for the existence of a symmetric networks with $2n$ nodes, in which each node has $n + \bar{\eta}$ links.

that deviates does not hold any interbank deposits. The lemma follows from the feasibility conditions introduced in Section 2.2. In other words, the lemma relies on the assumption that there are no defaults, and that a bank delivers at date 1 and 2 the deposit contract she had promised at date 0, in states S^1 and S^2. Given this, I show that there is no pattern of interbank deposits such that a bank that deviates from the contract (1) and the portfolio (2) exchanges positive amounts with other banks in the system. Otherwise, there must be at least another bank that defaults on the contract (1), either in state S^1 or S^2. Therefore, the bank that deviates must offer a deposit contract for early and late withdrawals, and hold a portfolio as when she is in autarky. This implies that the expected utility loss in states S^1 and S^2 for the bank that deviates is strictly positive. Thus, even though the bank may obtain a lower utility in state $\bar{S}$ when she offers the contract (1) and holds the portfolio (2) than when she deviates, the deviation is sub-optimal if the probability of the aggregate shock is sufficiently small.

Proposition 5 formalizes the reasoning outlined in Allen and Gale (2000). Although they do not explicitly write down the feasibility constraints at date 1 and 2, they argue that in order to avoid default, a bank has to make a large deviation, rather than a small one. The distortion this causes in the other states will not be worth it, if the probability of the aggregate shock is sufficiently small.

This result is indeed consistent with the numerical findings in Dasgupta (2004), who illustrates that it is optimal for banks to fully insure against regional liquidity shocks, as long as the probability of contagion is small.

5 Discussion

5.1 The model with linking costs

As described in Section 4, there is an implicit cost associated to forming links. In particular, a bank i must consider the utility loss for her late consumers that can potentially arise when forming a link with a bank j that has $\eta_j \geq \bar{\eta}$ links. This implicit cost introduces a natural trade-off against the benefits that a link can bring. However, there is no cost associated to forming a link with a bank that has less than $\bar{\eta}$ links. In fact, as Properties $7-8$ imply, there are many cases in which banks are indifferent between forming or severing links. In the previous section I assumed that when two banks are indifferent between forming and not forming the link, then they form the link, and when they are indifferent between keeping and severing a link, then they keep the link. However, it is interesting to understand what is the effect of introducing a small linking cost.

In this section I analyze the equilibrium interbank networks that arise when each bank incurs a linking cost, ε. Because links with banks that have at least $\bar{\eta}$ links are inherently costly, I am interested in the case when the cost of linking only marginally breaks the indifference in banks payoffs implied by Properties $7-8$. In other words, I consider that the linking cost is sufficiently small that the ordering of payoffs described in Properties $1-6$ does not change. However, if two banks i, with $\eta_i \leq \bar{\eta}-2$, and j, with $\eta_j \leq \bar{\eta}-2$, do not have a link in the network, they chose not to form a link. Similarly, if two banks i, with $\eta_i \leq \bar{\eta}-1$, and j, with $\eta_j \leq \bar{\eta}-1$, have a link in the network, they chose not to severe the link.

The existence of a cost of linking, ε, increases the set of pairwise stable interbank networks. For instance, a network in which banks have no links with other banks in the same region is pairwise stable. At the same time, a network in which all banks have $\bar{\eta}$ links is also pairwise stable. In fact, the number of banks that have at least $\bar{\eta}$ links with banks in the same region in a pairwise stable network can vary from zero to $2n$. Similarly, the probability that all banks default in state $\bar{S}$ ranges from 0 to φ.

Although in the presence of small linking costs it is not possible to provide an upper bound for the probability that all banks default, I can nevertheless provide a ranking of equilibria depending on their implied expected welfare. The result is similar to the one derived in Section 4.

Proposition 6 *Let* $r < \frac{z^*(n+\bar{\eta})}{n((2z^*+1)(n+\bar{\eta})-1)}$. *Then, there exists* $\bar{\varepsilon}$ *such that for any linking cost* $\varepsilon < \bar{\varepsilon}$, *the expected welfare in pairwise stable networks is increasing in the number of banks that have* $\bar{\eta}$ *links.*

It is interesting to compare the outcomes in the case when there is a small linking cost with the outcomes in the model in which banks favor linking, analyzed in the previous section. In both cases, the pairwise network that yields the highest welfare is a network in which all banks have $\bar{\eta}$ links. However, when links are costly there is no lower bound on the number of banks that have at least $\bar{\eta}$ links. In contrast, in the model in which banks favor linking there are at least $\max\{2(n-\bar{\eta}),n\}$ banks that have at least $\bar{\eta}$ links. As welfare is monotonic in both cases, this implies that the welfare in many equilibria in the model with linking costs is lower than the minimum welfare that can be supported in an equilibrium of the model in which banks favor linking.

This is interesting from a policy perspective as well. It appears that small transaction costs are detrimental to financial stability. Indeed, not only that there are more equilibria when links are costly, but there is also a higher risk that all banks default by contagion in the additional equilibria.

5.2 Welfare in interbank networks

The intuition developed in Allen and Gale (2000) is that complete networks are more resilient to contagion relative to incomplete networks. Inequality (13) shows indeed that if each bank has at least $\bar{\eta}$ links with banks in the same region there is no contagion in an interbank network. In fact, in any network in which all banks have at least $\bar{\eta}$ links there is no contagion in state $\bar{S}$. However, it is not necessarily the case that a network in which banks have more than $\bar{\eta}$ links improves consumers' welfare.

To understand how consumers' welfare depends on the number of links that each bank has with banks in the same region, I compare symmetric networks, $g_{n,\eta}$, in which each bank has $(n+\eta)$ links, for $\eta\in\{\bar{\eta},\bar{\eta}+1,...,n-1\}$. As for any $\eta\geq\bar{\eta}$ there is no contagion in an interbank network $g_{n,\eta}$, then in state $\bar{S}$ one bank is affected by the early-withdrawal shock, $(n+\eta)$ banks repay C_1^* to early consumers, but less than C_2^* to late consumers, and $(2n-(n+\eta+1))$ banks are able to repay (C_1^*,C_2^*) to depositors. Using(11) and (12), I obtain the expected welfare of consumers in state $\bar{S}$ as

$$\begin{aligned} W^{\bar{S}}\left(g_{n,\eta}\right) &= (2n-(n+\eta+1))\times\left(q\ln(C_1^*)+(1-q)\ln\left(C_2^*\right)\right) \\ &\quad +(n+\eta)\times\left(q\ln(C_1^*)+(1-q)\ln\left(C_2^*-\frac{z^*}{n}\frac{\frac{R}{r}C_1^*-C_2^*}{1+(n+\eta)\frac{z^*}{n}}\right)\right) \\ &\quad +\ln\left(C_1^*-\frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+\eta)\,\frac{z^*}{n}}\right). \end{aligned}$$

Two opposite effects are transparent from this expression. On the one hand, the consumers of the bank that is affected by the early-withdrawal shock and of her neighbors benefit

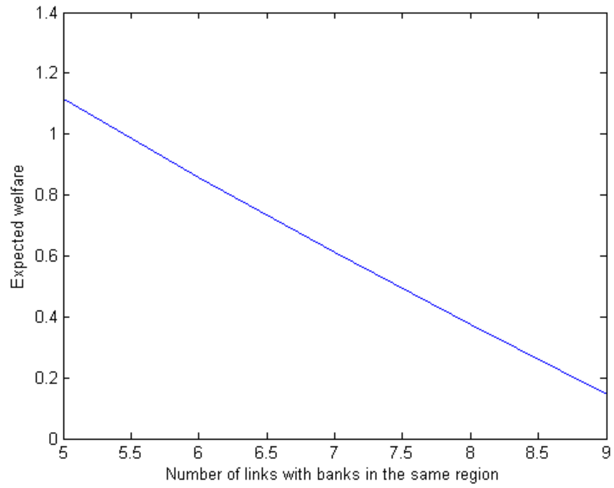

Figure 2: The figures depicts the expected welfare for an interbank network with $2n = 20$ banks in which contagion does not occur if each bank has at least $\bar{\eta} = 5$ links, as the number of link that each bank increases from 5 to $n - 1 = 9$. The parameters for this plot are as follows: $R = 2$, $r = 0.03$, $p_H = 0.6$, $p_L = 0.2$. Consumers are assumed to have log-preferences.

as η increases. On the other hand, the number of banks that are able to repay (C_1^*, C_2^*) to their depositors decreases, as η increases. Figure 2 illustrates that the second effects can dominate. In particular, the figures shows that the welfare for an interbank network with $2n = 20$ banks in which contagion does not occur if each bank has at least $\bar{\eta} = 5$ links decreases as the number of link that each bank increases from 5 to $n - 1 = 9$. Thus, equilibrium networks in which contagion does not occur can also have the highest associated welfare.

5.3 Incomplete liquidity networks, or $\ell < n$

The main analysis of equilibrium solvency networks is under the assumption that the liquidity network is complete. That is, each bank has links with all other banks in the other region ($\ell = n$, $\forall\, i \in N$). It would be interesting to understand however what effects arise when the liquidity network is incomplete, or when $\ell < n$.

Consider $\bar{\eta}(\ell)$ be the smallest integer for which inequality (10). Then, there are two possibilities when $\ell < n$. First, let ℓ be such that all banks default by contagion if the bank that is subject to the early-withdrawal shock has less than $\bar{\eta}(\ell)$ connections, for

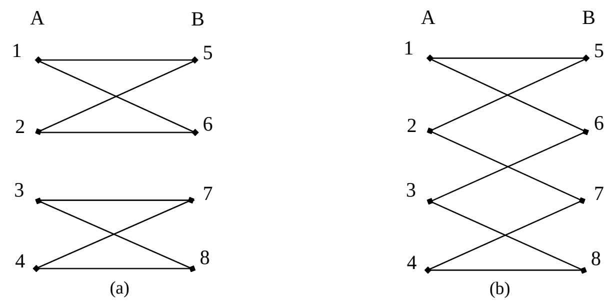

Figure 3: This figures illustrates two possible patterns of connection between banks, when each bank has $\ell = 2$ liquidity links with banks in the other region. Panel (a) illustrates a network with two disconnected components. Panel (b) illustrates a connected interbank network, in which from each bank there is a path of links to any other bank.

any pattern of links between banks or number of links with banks in the same region. In this case, the result in Proposition 1 holds and all banks return C_1^d per unit of deposit as given by (14). Then, properties $1-8$ described in Section 4 still characterize, at least qualitatively, the payoffs for forming or severing links with banks in the same region. In these conditions, it is possible to establish a lower bound on the number of banks that have $\bar{\eta}(\ell)$ links similar to the one in Proposition 3.

Second, consider the case when ℓ is such that not all banks default by contagion if the bank that is subject to the early-withdrawal shock has less than $\bar{\eta}(\ell)$ connections. A stark example is given by the case of a banking system with eight banks in which each banks has $\ell = 2$ links with banks in the other region and no links with banks in the same region. Suppose that $\bar{\eta}(2) = 1$. In Figure 3(a), the interbank network is disconnected in two separate components. Then, only four banks default when a bank is affected by an early-withdrawal shock. In contrast, in Figure 3(b), the interbank network is connected as from each bank there is a path of links to any other bank in the interbank network. Then, when a bank is affected by the early-withdrawal shock, her two immediate neighbors default, then her neighbors' neighbors default, with all banks defaulting at the end.[8]

As explained in Section 4, the payoffs that banks receive in the general case when

[8] Connectedness is not a sufficient condition for all banks to default. One can construct more intricate examples of connected networks in which the first wave of defaults is contained. For instance, suppose that the neighbors of a bank affected by the idiosyncratic shock are well linked. Then, even if they default, their neighbors can act as a buffer so contagion does not spread.

$\ell < n$ depend on the entire network structure. For instance, the expected payoff of bank 3 in the network in Figure 3(a) is different than her expected payoff in the network in Figure 3(b). In the former, bank 3 defaults by contagion only if the other three banks in her component are affected by the early-withdrawal shocks, whereas in the latter she defaults by contagion if any of the other seven banks in the system is affected by the early-withdrawal shock. Consequently, the marginal payoffs for forming or severing links also depend on the entire network structure. As before, banks are willing to incur a loss on their deposits if they can avoid defaulting by contagion. In our example, bank 3 has an incentive to form a link with bank 4, but not with bank 2 or 1. In other words, banks' free-riding on others links becomes more severe.

6 Conclusions

The problem of contagion within the banking system is an intensely debated issue. In this paper I develop a model that explains how interdependencies between banks emerge endogenously. In particular, I investigate how banks form links with each other, when the banking system is exposed to contagion risk. The question I address is wether banks form networks that are resilient to the propagation of small early-withdrawal shocks.

Banks internalize the risk of contagion and insure against it by forming links with other banks in the banking system. I study whether the stable network architectures that emerge support systemic stability. I show that, interbank networks that arise in equilibrium have different levels of financial stability. In certain equilibria contagion does not occur.

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A Appendix A: Derivations of the main results

Proof of Proposition1.

First, note that a positive integer $\bar{\eta} \leq n-1$ exists when

$$\frac{Rz}{(R-1)\,n\,(2z+1)+z^*} \leq r \leq \frac{Rz}{(R-1)\,n\,(1+z^*)+Rz}.$$

Then, consider g_{n,η_i} to be a network in which each bank i has connections with all the other n banks in the other region and η_i with banks in the same region.

Fix a bank i^* in region A (the argument is identical if $i^* \in B$). Suppose that bank i^* is affected by the early-withdrawal shock and consider that the number of connections that i^* has with banks in the same region, η_{i^*}, is such that

$$\frac{z^*}{n}\frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_{i^*})\frac{z^*}{n}} > r \cdot b(q).$$

Then the loss-given-default that i^* induces to a neighbor j is

$$LGD_{ji^*} > r \cdot b(q)$$

for any $j \in N_{i^*}\left(g_{n,\eta_i}\right)$. It follows that all of the $(n+\eta_{i^*})$ neighbors of bank i^* must liquidate too much of the illiquid asset, and default by contagion.

Consider the remaining set of banks $k \in N \setminus N_{i^*}\left(g_{n,\eta_i}\right)$. Each bank k is in region A, just as i^*, and must be connected with all the banks in region B. Each bank k must liquidate an amount of the illiquid asset of

$$\sum_{j \in N_k\left(g_{n,\eta_i}\right)} LGD_{jk} \geq \sum_{j \in B} \frac{z^*}{n}\left(\frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+\left(n+\eta_j\right)\cdot\frac{z^*}{n}} + \frac{\sum_{j' \in N_j\left(g_{n,\eta_i}\right)} \frac{z^*}{n}\left(C_1^* - C_{1j'}^d\right)}{1+\left(n+\eta_j\right)\cdot\frac{z^*}{n}}\right) \tag{17}$$

where I used (8) for $\ell = n$, and accounted only for the loss-given-default induced by banks in region B.

As $j \in N_{i^*}\left(g_{n,\eta_i}\right)$, for each $j \in B$ we have that

$$\sum\nolimits_{j' \in N_j\left(g_{n,\eta_i}\right)} \frac{z^*}{n}\left(C_1^* - C_{1j'}^d\right) \geq \frac{z^*}{n}\left(C_1^* - C_{1i^*}^d\right)$$

or

$$\sum\nolimits_{j' \in N_j(g_n)} \frac{z^*}{n}\left(C_1^* - C_{1j'}^d\right) > r \cdot b(q).$$

Using that $\eta_j < n$ for any j, we have that

$$\sum_{j \in B} \frac{z^*}{n} \left(\frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+\left(n+\eta_j\right)\cdot\frac{z^*}{n}} + \frac{\sum_{j' \in N_j\left(g_{n,\eta_i}\right)} \frac{z^*}{n}\left(C_1^* - C_{1j'}^d\right)}{1+\left(n+\eta_j\right)\cdot\frac{z^*}{n}} \right) \geq n\frac{z^*}{n}\left(\frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+2n\cdot\frac{z^*}{n}} + \frac{r\cdot b(q)}{1+2n\cdot\frac{z^*}{n}} \right). \tag{18}$$

As

$$\frac{z^*}{(1+z^*)}(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right) > \frac{z^*}{n}\frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_{i^*})\frac{z^*}{n}}$$

then

$$\frac{z^*}{(1+z^*)}(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right) > r\cdot b(q)$$

or

$$z^*\left((1-q)\left(C_1^* - \frac{r}{R}C_2^*\right) + r\cdot b(q)\right) > (1+2z)\cdot r\cdot b(q)$$

which implies that

$$n\frac{z^*}{n}\left(\frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+2n\cdot\frac{z^*}{n}} + \frac{r\cdot b(q)}{1+2n\cdot\frac{z^*}{n}} \right) > r\cdot b(q). \tag{19}$$

From (17), (18), and (19), it follows that

$$\sum_{j\in N_k\left(g_{n,\eta_i}\right)} LGD_{jk} > r\cdot b(q)$$

for any bank $k \in N \setminus N_{i^*}\left(g_{n,\eta_i}\right)$. This implies that if i^* is affected by the early-withdrawal shock, then all other banks fails by contagion in the network g_{n,η_i}.

From (6) we obtain that any bank i return in period 1

$$C_{1i}^d = \frac{x^* + ry^* + \sum_{k\in N_i\left(g_{n,\eta_i}\right)} \frac{z^*}{n} C_{1k}^d}{1+(n+\eta_i)\cdot\frac{z^*}{n}}.$$

This represents a linear system of $2n$ equations with $2n$ unknowns. The vector of returns $(C_{1i})_{i\in A\cup B}$ that each bank pays when defaulting is a fixed point of this system. I search for a symmetric solution, in which all banks return C_1^d when they default. In other words C_1^d must satisfy

$$C_1^d = \frac{x^* + ry^* + \sum_{k\in N_i\left(g_{n,\eta_i}\right)} \frac{z^*}{n} C_1^d}{1+(n+\eta_i)\cdot\frac{z^*}{n}}$$

or

$$C_1^d = \frac{x^* + ry^* + (n+\eta_i)\cdot\frac{z^*}{n} C_1^d}{1+(n+\eta_i)\cdot\frac{z^*}{n}}.$$

It follows that

$$C_1^d = x^* + ry^*.$$

■

Proof of Proposition 3.

Case 1: If $\bar{\eta} < \frac{n}{2}$, then there are at least $2(n - \bar{\eta})$ banks that have at least $\bar{\eta}$ links with banks in the same region.

Consider the two regions in the banking system $A = \{1, 2, ..., n\}$ and $B = \{n+1, n = 2, ..., 2n\}$. Let $H(A) = \{i \in A \,|\eta_i \geq \bar{\eta}\}$ and $H(B) = \{i \in B \,|\eta_i \geq \bar{\eta}\}$. In order to prove that there are least $2(n - \bar{\eta})$ banks that have $\bar{\eta}$ links, I show that $|H(A)| \geq n - \bar{\eta}$ and $|H(B)| \geq n - \bar{\eta}$. Because the cases are symmetric, I only prove that $|H(A)| \geq n - \bar{\eta}$.

Suppose the contrary, that $|H(A)| < n - \bar{\eta}$. This implies that $|A \setminus H(A)| \geq n - (n - \bar{\eta} - 1)$. In other words, the set of banks that have at most $\bar{\eta} - 1$ links has at least $\bar{\eta} + 1$ elements. Properties $3-4$ and $7-8$ imply that in a stable network all banks with $\eta_i \leq \bar{\eta} - 1$ must be directly linked with each other. Thus, each bank in $A \setminus H(A)$ has links at least with all other banks in $A \setminus H(A)$. It follows that each bank in $A \setminus H(A)$ has at least $\bar{\eta}$ links. We arrived thus to a contradiction.

Case 2: If $\bar{\eta} \geq \frac{n}{2}$, then there are at least n banks that have at least $\bar{\eta}$ links with banks in the same region.

As before, I only prove that $|H(A)| \geq n/2$.

Let $|H(A)| = h_A$. If $h_A \geq \bar{\eta}$, the proof is complete.

Consider the case when $h_A < \bar{\eta}$. Properties $3 - 4$ and $7 - 8$ imply that in a stable network all banks with $\eta_i \leq \bar{\eta} - 1$ must be directly linked with each other. Thus, each bank in $A \setminus H(A)$ has links at least with all other banks in $A \setminus H(A)$. This implies that the total number of links[9] between banks in region A that have $\eta_i \leq \bar{\eta} - 1$ must be $(n - h_A)(n - h_A - 1)$. In addition, since $h_A < \bar{\eta}$, it must be that each bank in $H(A)$ has some links with banks in $A \setminus H(A)$. Assuming that all banks in $H(A)$ are directly linked with each other, there must be at least $h_A(\bar{\eta} - h_A + 1)$ links with banks in $A \setminus H(A)$.

As all the banks in $A \setminus H(A)$ have at most $\bar{\eta} - 1$ links, the total amount of links these

[9] Links here are counted twice for each node. However, I double counte all links for the rest of the proof, such that in the end it cancels out.

banks have should not exceed $(n-h_A)\bar{\eta}$. Thus, the following inequality must hold:

$$(n-h_A)(n-h_A-1)+h_A(\bar{\eta}-h_A+1)<(n-h_A)\bar{\eta}$$

This inequality can be rewritten as

$$(\bar{\eta}-h_A+1)(2h_A-n)<(n-h_A)(2h_A-n)$$

As $\bar{\eta}-h_A+1<n-h_A$, it must be that $2h_A-n>0 \Leftrightarrow h_A>n/2$. This is a contradictions and concludes the proof. ■

Proof of Proposition 4.

Suppose that $r<\frac{z^*(n+\bar{\eta})}{n((2z+1)(n+\bar{\eta})-1)}$. As in Section 3, let $H(g_{n,\eta_i})=\{j\in A\cup B|\eta_j=\bar{\eta}\}$ be the set of banks that have $\bar{\eta}$ links, and let $h=\left|H(g_{n,\eta_i})\right|$ be the number of banks that have $\bar{\eta}$ links. Whenever a bank $j\in H(g_{n,\eta_i})$ is affected by the early-withdrawal shock, each neighbor i of j returns to the late consumers

$$C_{2j}^d(\bar{\eta})=C_2^*-\frac{z^*}{n}\frac{\frac{R}{r}C_1^*-C_2^*}{1+(n+\bar{\eta})\frac{z^*}{n}},$$

while i returns

$$C_{1i}^d(\bar{\eta})=C_1^*-\frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+\bar{\eta})\frac{z^*}{n}}.$$

However, when any of the other $(2n-h)$ banks is affected by the early-withdrawal shock all banks default by contagion, and return

$$C_1^d=C_1^*-(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right).$$

The expected welfare in a network g_{n,η_i} in which there are h banks that have $\bar{\eta}$ links and $(2n-h)$ banks that have less then $\bar{\eta}$ links is given by

$$\begin{aligned}W(g_{n,\eta_i}) &= (1-\varphi)\left\{2n\times\left[qu\left(C_1^*\right)+(1-q)\,u\left(C_2^*\right)\right]\right\}\\ &\quad+\varphi\left\{\frac{2n-h}{2n}\left[2n\times u\left(C_1^*-(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)\right)\right]\right.\\ &\quad+\frac{h}{2n}\left[(2n-(n+\bar{\eta}+1))\times(qu(C_1^*)+(1-q)\,u\left(C_2^*\right))+u\left(C_1^*-\frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+\bar{\eta})\frac{z^*}{n}}\right)\right.\\ &\quad\left.\left.+(n+\bar{\eta})\times\left(qu(C_1^*)+(1-q)\,u\left(C_2^*-\frac{z^*}{n}\frac{\frac{R}{r}C_1^*-C_2^*}{1+(n+\bar{\eta})\frac{z^*}{n}}\right)\right)\right]\right\}\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial}{\partial h}W &= \varphi\Big\{-u\left(C_1^* - (1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)\right) \\ &+\frac{1}{2n}\left[(2n-(n+\bar{\eta}+1))\times\left(qu(C_1^*)+(1-q)\,u\left(C_2^*\right)\right)+u\left(C_1^* - \frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+\bar{\eta})\,\frac{z^*}{n}}\right)+\right. \\ &\left.+(n+\bar{\eta})\times\left(qu(C_1^*)+(1-q)\,u\left(C_2^*-\frac{z^*}{n}\frac{\frac{R}{r}C_1^*-C_2^*}{1+(n+\bar{\eta})\frac{z^*}{n}}\right)\right)\right]\Big\}\end{aligned}$$

As

$$qu(C_1^*)+(1-q)\,u\left(C_2^*\right) > u\left(C_1^* - (1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)\right),$$

and

$$qu(C_1^*)+(1-q)\,u\left(C_2^*-\frac{z^*}{n}\frac{\frac{R}{r}C_1^*-C_2^*}{1+(n+\bar{\eta})\frac{z^*}{n}}\right) > u\left(C_1^* - (1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)\right),$$

and

$$u\left(C_1^* - \frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+\bar{\eta})\,\frac{z^*}{n}}\right) > u\left(C_1^* - (1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)\right),$$

it follows that

$$\frac{\partial}{\partial h}W > 0$$

■

Lemma 1 *Let*

$\mathcal{C}_i\left(A\right)=\left\{\left(C_{1i}^S, C_{2i}^S\right),\left(x_i,y_i\right)\,\middle|\, p_H C_{1i}^{S^1} = p_L C_{1i}^{S^2} = x_i \text{ and } \left(1-p_H\right)C_{2i}^{S^1} = \left(1-p_L\right)C_{2i}^{S^2} = Ry_i\right\}$

and

$\mathcal{C}_i\left(B\right)=\left\{\left(C_{1i}^S, C_{2i}^S\right),\left(x_i,y_i\right)\,\middle|\, p_L C_{1i}^{S^1} = p_H C_{1i}^{S^2} = x_i \text{ and } \left(1-p_L\right)C_{2i}^{S^1} = \left(1-p_H\right)C_{2i}^{S^2} = Ry_i\right\}$.

Then, the set of feasible contracts and portfolios that bank i can offer when all other banks $j \in A\cup B\setminus\{i\}$ offer the deposit contract (C_1^*, C_2^*) and hold the portfolio (x^*, y^*) is given by

$$\mathcal{C}_i\left(A\right)\cup\left\{\left(C_1^*, C_2^*\right),\left(x^*, y^*\right)\right\}$$

if $i \in A$, and

$$\mathcal{C}_i\left(B\right)\cup\left\{\left(C_1^*, C_2^*\right),\left(x^*, y^*\right)\right\}$$

if $i \in B$.

Proof. The derivations are provided for the case when $i \in A$. The case when $i \in B$ follows by symmetry and is omitted.

It is straightforward that the contract (C_1^*, C_2^*) and the portfolio (x^*, y^*) are feasible. Next I consider contracts $\left(C_{1i}^S, C_{2i}^S\right)$ and portfolios (x_i, y_i) that are different then (C_1^*, C_2^*) and (x^*, y^*), and check which ones respect the feasibility constraints.

I first introduce some notation in order to express the feasibility constraints in matrix form.

Let $\mathbf{x}_A = (x_i)_{i\in A}$, $\mathbf{x}_B = (x_i)_{i\in B}$, $\mathbf{y}_A = (y_i)_{i\in A}$ and $\mathbf{y}_B = (y_i)_{i\in A}$ be $n \times 1$ column vectors representing the liquid and illiquid asset holdings of banks in region A and B, respectively.

Let $Z_A = (z_{ij})_{\substack{i\in A\\ j\in A}}$ be the $n \times n$ matrix of interbank deposits that banks in region A hold at other banks in region A, with $z_{ij} = 0$ if $i = j$ or if i and j do not have a link. Similarly, I define $Z_B = (z_{ij})_{\substack{i\in B\\ j\in B}}$ be the $n \times n$ matrix of interbank deposits that banks in region B hold at other banks in region B, with $z_{ij} = 0$ if $i = j$ or if i and j do not have a link. Let $Z_{AB} = (z_{ij})_{\substack{i\in A\\ j\in B}}$ be the $n \times n$ matrix of interbank deposits that banks in region A hold at banks in region B, and $Z_{BA} = (z_{ij})_{\substack{i\in B\\ j\in A}}$ be the $n \times n$ matrix of interbank deposits that banks in region B hold at banks in region A, with $z_{ij} = 0$ if i and j do not have a link.

Let $\mathbf{1}_i$ be the $n \times 1$ column vector that has all elements 0, except for the ith element which is equal to 1. Let $\mathbf{1}_{i\times i}$ be the $n \times n$ matrix with all elements 0, except for row i which has all elements equal to 1. The identity matrix is denoted by I, while I_i represents the identity matrix with all elements 0 on row i.

Then, when bank i offers a contract $\left(C_{1i}^S, C_{2i}^S\right)$ and holds a portfolio (x_i, y_i) while all other banks $j \in A \cup B \setminus \{i\}$ offer the deposit contract (C_1^*, C_2^*) and hold the portfolio (x^*, y^*), the feasibility constraints can be written for each date, state by state, as follows.

The date 0 feasibility constraint for banks in region A is

$$\mathbf{x}_A + \mathbf{y}_A + Z_A \cdot \mathbf{1} + Z_{AB} \cdot \mathbf{1} = \mathbf{1} + Z_A^T \cdot \mathbf{1} + Z_{BA}^T \cdot \mathbf{1}, \tag{20}$$

while for banks in region B is

$$\mathbf{x}_B + \mathbf{y}_B + Z_B \cdot \mathbf{1} + Z_{BA} \cdot \mathbf{1} = \mathbf{1} + Z_B^T \cdot \mathbf{1} + Z_{AB}^T \cdot \mathbf{1}. \tag{21}$$

The date 1 feasibility constraint for banks in region A is

$$\begin{aligned}\mathbf{x}_A + C_1^* \cdot Z_A \cdot \mathbf{1} + C_1^* \cdot Z_{AB} \cdot \mathbf{1} + \left(C_{1i}^{S_1} - C_1^*\right) \cdot Z_A \cdot \mathbf{1}_i &= p_H C_1^* \cdot \mathbf{1} + p_H \left(C_{1i}^{S_1} - C_1^*\right) \cdot \mathbf{1}_i + \\ &\quad C_1^* \cdot Z_A^T \cdot \mathbf{1} + \left(C_{1i}^{S_1} - C_1^*\right) \cdot \mathbf{1}_{i\times i} \cdot Z_A \cdot \mathbf{1}_i \end{aligned} \tag{22}$$

in state S^1, and

$$\begin{aligned}\mathbf{x}_A + C_1^* \cdot Z_A \cdot \mathbf{1} + \left(C_{1i}^{S_2} - C_1^*\right) \cdot Z_A \cdot \mathbf{1}_i &= p_L C_1^* \cdot \mathbf{1} + p_L \left(C_{1i}^{S_2} - C_1^*\right) \cdot \mathbf{1}_i + \\ &\quad C_1^* \cdot Z_A^T \cdot \mathbf{1} + C_1^* \cdot Z_{BA}^T \cdot \mathbf{1} + \\ &\quad \left(C_{1i}^{S_2} - C_1^*\right) \cdot \mathbf{1}_{i\times i} \cdot Z_A \cdot \mathbf{1}_i + \left(C_{1i}^{S_2} - C_1^*\right) \cdot \mathbf{1}_{i\times i} \cdot Z_{BA} \cdot \mathbf{1}_i \end{aligned} \tag{23}$$

in state S^2.

The date 1 feasibility constraint for banks in region B is

$$\mathbf{x}_B + C_1^* \cdot Z_B \cdot \mathbf{1} = p_L C_1^* \cdot \mathbf{1} + C_1^* \cdot Z_B^T \cdot \mathbf{1} + C_1^* \cdot Z_{AB}^T \cdot \mathbf{1} \tag{24}$$

in state S^1, and

$$\mathbf{x}_B + C_1^* \cdot Z_B \cdot \mathbf{1} + C_1^* \cdot Z_{BA} \cdot \mathbf{1} + \left(C_{1i}^{S_2} - C_1^*\right) \cdot Z_{BA} \cdot \mathbf{1}_i = p_H C_1^* \cdot \mathbf{1} + C_1^* \cdot Z_B^T \cdot \mathbf{1} \tag{25}$$

in state S^2.

The date 2 feasibility constraint for banks in region A is

$$R\mathbf{y}_A = (1 - p_H)\, C_2^* \cdot \mathbf{1} + (1 - p_H) \left(C_{2i}^{S_1} - C_2^*\right) \cdot \mathbf{1}_i + C_2^* \cdot Z_{BA}^T \cdot \mathbf{1} + \left(C_{2i}^{S_1} - C_2^*\right) \cdot \mathbf{1}_{i\times i} \cdot Z_{BA} \cdot \mathbf{1}_i \tag{26}$$

in state S^1, and

$$R\mathbf{y}_A + C_2^* \cdot Z_{AB} \cdot \mathbf{1} = (1 - p_L)\, C_2^* \cdot \mathbf{1} + (1 - p_L) \left(C_{2i}^{S_2} - C_2^*\right) \cdot \mathbf{1}_i \tag{27}$$

in state S^2.

The date 1 feasibility constraint for banks in region B is

$$R\mathbf{y}_B + C_2^* \cdot Z_{BA} \cdot \mathbf{1} + \left(C_{2i}^{S_1} - C_2^*\right) \cdot Z_{BA} \cdot \mathbf{1}_i = (1 - p_L)\, C_2^* \cdot \mathbf{1} \tag{28}$$

in state S^1, and

$$R\mathbf{y}_B = (1 - p_H)\, C_2^* \cdot \mathbf{1} + C_2^* \cdot Z_{AB}^T \cdot \mathbf{1} \tag{29}$$

in state S^2.

- **Step 1:** I first show that $Z_{BA} \cdot \mathbf{1}_i = 0$.

  Substracting (24) from (25), we obtain

$$Z_{BA} \cdot \mathbf{1} + Z_{AB}^T \cdot \mathbf{1} = (p_H - p_L) \cdot \mathbf{1} - \frac{\left(C_{1i}^{S_2} - C_1^*\right)}{C_1^*} \cdot Z_{BA} \cdot \mathbf{1}_i$$

  and, substracting (29) from (28), we obtain

$$Z_{BA} \cdot \mathbf{1} + Z_{AB}^T \cdot \mathbf{1} = (p_H - p_L) \cdot \mathbf{1} - \frac{\left(C_{2i}^{S_1} - C_2^*\right)}{C_2^*} \cdot Z_{BA} \cdot \mathbf{1}_i$$

  which implies that

$$\left(\frac{\left(C_{2i}^{S_1} - C_2^*\right)}{C_2^*} - \frac{\left(C_{1i}^{S_2} - C_1^*\right)}{C_1^*} \right) \cdot Z_{BA} \cdot \mathbf{1}_i = 0 \tag{30}$$

  In addition, adding up (24) and (25), we obtain that

$$2\mathbf{x}_B + 2C_1^* \cdot Z_B \cdot \mathbf{1} + C_1^* \cdot Z_{BA} \cdot \mathbf{1} + \left(C_{1i}^{S_2} - C_1^*\right) \cdot Z_{BA} \cdot \mathbf{1}_i = (p_H + p_L)\, C_1^* \cdot \mathbf{1} + 2C_1^* \cdot Z_B^T \cdot \mathbf{1} + C_1^* \cdot Z_{AB}^T \cdot \mathbf{1}.$$

  Using that $\mathbf{x}_B = qC_1^* \cdot \mathbf{1}$, and substituting $q = \frac{p_H + p_L}{2}$, we have

$$2 \cdot Z_B \cdot \mathbf{1} + Z_{BA} \cdot \mathbf{1} + \frac{\left(C_{1i}^{S_2} - C_1^*\right)}{C_1^*} \cdot Z_{BA} \cdot \mathbf{1}_i = 2 \cdot Z_B^T \cdot \mathbf{1} + Z_{AB}^T \cdot \mathbf{1}. \tag{31}$$

  Similarly, adding up (29) and (28), we obtain that

$$2R\mathbf{y}_B + C_2^* \cdot Z_{BA} \cdot \mathbf{1} + \left(C_{2i}^{S_1} - C_2^*\right) \cdot Z_{BA} \cdot \mathbf{1}_i = (2 - p_L - p_H)\, C_2^* \cdot \mathbf{1} + C_2^* \cdot Z_{AB}^T \cdot \mathbf{1}.$$

  Using that $\mathbf{y}_B = \frac{(1-q)C_2^*}{R} \cdot \mathbf{1}$, and substituting $q = \frac{p_H + p_L}{2}$, we have

$$Z_{BA} \cdot \mathbf{1} + \frac{\left(C_{2i}^{S_1} - C_2^*\right)}{C_2^*} \cdot Z_{BA} \cdot \mathbf{1}_i = Z_{AB}^T \cdot \mathbf{1} \tag{32}$$

  Adding up (31) and (32), we obtain that

$$2 \cdot Z_B \cdot \mathbf{1} + 2 \cdot Z_{BA} \cdot \mathbf{1} + \frac{\left(C_{1i}^{S_2} - C_1^*\right)}{C_1^*} \cdot Z_{BA} \cdot \mathbf{1}_i + \frac{\left(C_{2i}^{S_1} - C_2^*\right)}{C_2^*} \cdot Z_{BA} \cdot \mathbf{1}_i = 2 \cdot Z_B^T \cdot \mathbf{1} + 2 \cdot Z_{AB}^T \cdot \mathbf{1}$$

  and since the date 0 feasibility constraint (21) implies that

$$Z_B \cdot \mathbf{1} + Z_{BA} \cdot \mathbf{1} = Z_B^T \cdot \mathbf{1} + Z_{AB}^T \cdot \mathbf{1}$$

as $\mathbf{x}_B + \mathbf{y}_B = \mathbf{1}$, then we have that

$$\left(\frac{\left(C_{2i}^{S_1} - C_2^* \right)}{C_2^*} + \frac{\left(C_{1i}^{S_2} - C_1^* \right)}{C_1^*} \right) \cdot Z_{BA} \cdot \mathbf{1}_i = 0 \tag{33}$$

From (30) and (33), it follows that

$$Z_{BA} \cdot \mathbf{1}_i = 0 \tag{34}$$

- **Step 2:** I show that $Z_A \cdot \mathbf{1}_i = 0$

  Substracting (23) from (22) we obtain that

$$\begin{aligned} Z_{AB} \cdot \mathbf{1} + Z_{BA}^T \cdot \mathbf{1} \;\; = \;\; & (p_H - p_L) \cdot \mathbf{1} + \frac{\left(p_H \left(C_{1i}^{S_1} - C_1^* \right) - p_L \left(C_{1i}^{S_2} - C_1^* \right) \right)}{C_1^*} \cdot \mathbf{1}_i + \\ & \frac{\left(C_{1i}^{S_1} - C_{1i}^{S_2} \right)}{C_1^*} \cdot \left(\mathbf{1}_{i \times i} - I \right) \cdot Z_A \cdot \mathbf{1}_i - \frac{\left(C_{1i}^{S_2} - C_1^* \right)}{C_1^*} \cdot \mathbf{1}_{i \times i} \cdot Z_{BA} \cdot \mathbf{1}_i, \end{aligned}$$

  and, substracting (27) from (26) we obtain that

$$\begin{aligned} Z_{AB} \cdot \mathbf{1} + Z_{BA}^T \cdot \mathbf{1} \;\; = \;\; & (p_H - p_L) \cdot \mathbf{1} + \frac{\left((1 - p_L) \left(C_{2i}^{S_2} - C_2^* \right) - (1 - p_H) \left(C_{2i}^{S_1} - C_2^* \right) \right)}{C_2^*} \cdot \mathbf{1}_i - \\ & \frac{\left(C_{2i}^{S_1} - C_2^* \right)}{C_2^*} \cdot \mathbf{1}_{i \times i} \cdot Z_{BA} \cdot \mathbf{1}_i \end{aligned}$$

  Using (34), this implies that

$$\left(p_H \frac{C_{1i}^{S_1}}{C_1^*} - p_L \frac{C_{1i}^{S_2}}{C_1^*} - (1 - p_L) \frac{C_{2i}^{S_2}}{C_2^*} + (1 - p_H) \frac{C_{2i}^{S_1}}{C_2^*} \right) \cdot \mathbf{1}_i + \frac{\left(C_{1i}^{S_1} - C_{1i}^{S_2} \right)}{C_1^*} \cdot \left(\mathbf{1}_{i \times i} - I \right) \cdot Z_A \cdot \mathbf{1}_i = 0$$

  which yields

$$Z_A \cdot \mathbf{1}_i = 0 \tag{35}$$

- **Step 3**: I show that $\mathbf{1}^T \cdot Z_A^T \cdot \mathbf{1}_i = 0$.

  Using (34) and (35) in (22) and (23), respectively, and adding them up, we obtain

$$\begin{aligned} 2\mathbf{x}_A + 2C_1^* \cdot Z_A \cdot \mathbf{1} + C_1^* \cdot Z_{AB} \cdot \mathbf{1} \;\; = \;\; & (p_H + p_L) \, C_1^* \cdot \mathbf{1} + \left(p_H \left(C_{1i}^{S_1} - C_1^* \right) + p_L \left(C_{1i}^{S_2} - C_1^* \right) \right) \cdot \mathbf{1}_i + \\ & 2C_1^* \cdot Z_A^T \cdot \mathbf{1} + C_1^* \cdot Z_{BA}^T \cdot \mathbf{1}. \end{aligned}$$

If we multiply this identity to the left both side with I_i, and make use that $I_i \cdot \mathbf{x}_A = qC_1^* \cdot I_i \cdot \mathbf{1}$, we have that

$$2C_1^* \cdot I_i \cdot Z_A \cdot \mathbf{1} + C_1^* \cdot I_i \cdot Z_{AB} \cdot \mathbf{1} = 2C_1^* \cdot I_i \cdot Z_A^T \cdot \mathbf{1} + C_1^* \cdot I_i \cdot Z_{BA}^T \cdot \mathbf{1}.$$

Further, multiplying to the left both side of (20) with I_i, and making use that $I_i \cdot (\mathbf{x}_A + \mathbf{y}_A) = I_i \cdot \mathbf{1}$, we obtain that

$$I_i \cdot Z_A \cdot \mathbf{1} + I_i \cdot Z_{AB} \cdot \mathbf{1} = I_i \cdot Z_A^T \cdot \mathbf{1} + I_i \cdot Z_{BA}^T \cdot \mathbf{1}, \tag{36}$$

which implies, from the identity above, that

$$C_1^* \cdot I_i \cdot Z_A \cdot \mathbf{1} = C_1^* \cdot I_i \cdot Z_A^T \cdot \mathbf{1}.$$

We can simplify C_1^* and multiply on both side with $\mathbf{1}^T$, and obtain

$$\mathbf{1}^T \cdot I_i \cdot Z_A \cdot \mathbf{1} = \mathbf{1}^T \cdot I_i \cdot Z_A^T \cdot \mathbf{1} \tag{37}$$

or, since $\mathbf{1}^T \cdot I_i = (\mathbf{1} - \mathbf{1}_i)^T$,

$$(\mathbf{1} - \mathbf{1}_i)^T \cdot Z_A \cdot \mathbf{1} = (\mathbf{1} - \mathbf{1}_i)^T \cdot Z_A^T \cdot \mathbf{1}.$$

As $\mathbf{1}^T \cdot Z_A \cdot \mathbf{1} = \mathbf{1}^T \cdot Z_A^T \cdot \mathbf{1}$, it must be that

$$\mathbf{1}_i^T \cdot Z_A \cdot \mathbf{1} = \mathbf{1}_i^T \cdot Z_A^T \cdot \mathbf{1},$$

which, from (35), implies that

$$\left(\mathbf{1}_i^T \cdot Z_A \cdot \mathbf{1}\right)^T = 0 \tag{38}$$

- **Step 4**: I show that $\mathbf{1}^T \cdot Z_{AB}^T \cdot \mathbf{1}_i = 0$

  Using (34) in (32), we obtain that

$$Z_{BA} \cdot \mathbf{1} = Z_{AB}^T \cdot \mathbf{1}.$$

The equations (36) and (37) imply that

$$I_i \cdot Z_{AB} \cdot \mathbf{1} = I_i \cdot Z_{BA}^T \cdot \mathbf{1}.$$

Adding-up these two identities, and multiplying to the left with $\mathbf{1}^T$, we obtain

$$\mathbf{1}^T \cdot Z_{BA} \cdot \mathbf{1} + \mathbf{1}^T \cdot I_i \cdot Z_{AB} \cdot \mathbf{1} = \mathbf{1}^T \cdot Z_{AB}^T \cdot \mathbf{1} + \mathbf{1}^T \cdot I_i \cdot Z_{BA}^T \cdot \mathbf{1}$$

or, since $\mathbf{1}^T \cdot I_i = \mathbf{1}^T - \mathbf{1}_i^T$,

$$\mathbf{1}^T \cdot (Z_{BA} + Z_{AB}) \cdot \mathbf{1} - \mathbf{1}_i^T \cdot Z_{AB} \cdot \mathbf{1} = \mathbf{1}^T \cdot \left(Z_{BA}^T + Z_{AB}^T\right) \cdot \mathbf{1} - \mathbf{1}_i^T \cdot Z_{BA}^T \cdot \mathbf{1}.$$

As $\mathbf{1}^T \cdot (Z_{BA} + Z_{AB}) \cdot \mathbf{1} = \mathbf{1}^T \cdot \left(Z_{BA}^T + Z_{AB}^T\right) \cdot \mathbf{1}$, then it follows that

$$\mathbf{1}_i^T \cdot Z_{AB} \cdot \mathbf{1} = \mathbf{1}_i^T \cdot Z_{BA}^T \cdot \mathbf{1},$$

which, from (34), implies that

$$\left(\mathbf{1}_i^T \cdot Z_{AB} \cdot \mathbf{1}\right)^T = 0. \tag{39}$$

The equations (34), (35), (38) and (39) imply that when bank i offers a contract $\left(C_{1i}^S, C_{2i}^S\right)$ and holds a portfolio (x_i, y_i) while all other banks $j \in A \cup B \setminus \{i\}$ offer the deposit contract (C_1^*, C_2^*) and hold the portfolio (x^*, y^*), then bank i cannot hold any interbank deposits at other banks, and other banks do not cannot hold interbank deposits at bank i

$$z_{ij} = 0 \text{ and } z_{ji} = 0,\ \forall j \in A \cup B.$$

The feasibility constraints for bank i become

$$x_i + y_i = 1$$
$$x_i = p_H C_{1i}^{S^1} = p_L C_{1i}^{S^2}$$
$$R y_i = (1 - p_H)\, C_{2i}^{S^1} = (1 - p_L)\, C_{2i}^{S^2}.$$

■

Proof of Proposition 5.

To prove the result I show that there exists $\bar{\varphi} \in (0, 1)$ such that bank i's best response is to offer the deposit contract (1) and the portfolio (2) if all the other banks $j \neq i$ offer the deposit contract (1) and the portfolio (2), for any $\varphi \leq \bar{\varphi}$ and for any $i \in A \cup B$.

The proof follows a similar argument as in the proof of Proposition 1 in Mailath, Postlewaite and Samuelson (2005). The derivations are provided for the case when $i \in A$. The case when $i \in B$ follows by symmetry and is omitted.

From Lemma 1 above, we know that the set of feasible contracts and portfolios that bank i can offer when all other banks $j \in A \cup B \setminus \{i\}$ offer the deposit contract (C_1^*, C_2^*) and hold the portfolio (x^*, y^*) is given by

$$\mathcal{C}_i(A) \cup \{(C_1^*, C_2^*), (x^*, y^*)\}.$$

Note that the result in Lemma 1 does not rely on any assumption about the pattern of interbank deposits between banks.

Then, I need to show that there exists $\bar{\varphi} \in (0,1)$ such that

$$\pi_i(g; (C_1^*, C_2^*), (x^*, y^*)) > \pi_i\left(\tilde{g}; \left(\tilde{C}_{1i}^S, \tilde{C}_{2i}^S\right), (\tilde{x}_i, \tilde{y}_i)\right)$$

for any $\varphi \leq \bar{\varphi}$, for any $\left(\tilde{C}_{1i}^S, \tilde{C}_{2i}^S\right), (\tilde{x}_i, \tilde{y}_i) \in \mathcal{C}_A$, and for any g and $\tilde{g}$. Here I allow for the possibility that a network $\tilde{g} \neq g$ arises on the continuation path induced by the contract $\left(\tilde{C}_{1i}^S, \tilde{C}_{2i}^S\right)$ and portfolio $(\tilde{x}_i, \tilde{y}_i)$. The payoffs are given by

$$\pi_i(g; (C_1^*, C_2^*), (x^*, y^*)) = (1-\varphi)\left[qu(C_1^*) + (1-q)\, u(C_2^*)\right] + \varphi u_i^g,$$

where u_i^g is the utility bank i obtains in the network g in state $\bar{S}$, and

$$\begin{aligned}\pi_i\left(\tilde{g}; \left(\tilde{C}_{1i}^S, \tilde{C}_{2i}^S\right), (\tilde{x}_i, \tilde{y}_i)\right) &= (1-\varphi)\left[\frac{1}{2}\left(p_H u(\tilde{C}_{1i}^{S^1}) + (1-p_H)\, u(\tilde{C}_{2i}^{S^1})\right) + \frac{1}{2}\left(p_L u(\tilde{C}_{1i}^{S^2}) + (1-p_L)\, u(\tilde{C}_{2i}^{S^2})\right)\right] \\ &\quad + \varphi u_i^{\tilde{g}},\end{aligned}$$

where $u_i^{\tilde{g}}$ is the utility bank i obtains in the network $\tilde{g}$ in state $\bar{S}$,

Consider the function

$$f(t) = \frac{1}{2}\left(p_H \ln t + (1-p_H)\ln\left(\frac{R(1-p_H t)}{(1-p_H)}\right)\right) + \frac{1}{2}\left(p_L \ln\left(\frac{p_H}{p_L}t\right) + (1-p_L)\ln\left(\frac{R(1-p_H t)}{(1-p_L)}\right)\right).$$

The function f is constructed such that

$$f\left(\tilde{C}_{1i}^{S^1}\right) = \frac{1}{2}\left(p_H u(\tilde{C}_{1i}^{S^1}) + (1-p_H)\, u(\tilde{C}_{2i}^{S^1})\right) + \frac{1}{2}\left(p_L u(\tilde{C}_{1i}^{S^2}) + (1-p_L)\, u(\tilde{C}_{2i}^{S^2})\right),$$

for any $\left(\tilde{C}_{1i}^S, \tilde{C}_{2i}^S\right), (\tilde{x}_i, \tilde{y}_i) \in \mathcal{C}_i(A)$. It is straightforward to show that[10]

$$\max_t f(t) = f\left(\frac{p_H + p_L}{2p_H}\right) = f\left(\frac{q}{p_H}\right)$$

[10] This follows from setting $\frac{\partial f}{\partial t} = 0$. Note that $t = \frac{p_H + p_L}{2p_H}$ is indeed a maximum as $\left.\frac{\partial^2 f}{\partial t^2}\right|_{t = \frac{p_H + p_L}{2p_H}} = -4\frac{p_H^2}{(2-p_H-p_L)(p_H+p_L)} < 0$.

where

$$f\left(\frac{q}{p_H}\right) = \frac{1}{2}\left((1-p_H)\ln R\frac{(1-q)}{1-p_H} + p_H \ln\frac{q}{p_H}\right) + \frac{1}{2}\left((1-p_L)\ln R\frac{(1-q)}{1-p_L} + p_L \ln\frac{q}{p_L}\right)$$

It follows that

$$f\left(\frac{q}{p_H}\right) \geq \frac{1}{2}\left(p_H u(\tilde{C}_{1i}^{S^1}) + (1-p_H)\,u(\tilde{C}_{2i}^{S^1})\right) + \frac{1}{2}\left(p_L u(\tilde{C}_{1i}^{S^2}) + (1-p_L)\,u(\tilde{C}_{2i}^{S^2})\right),$$

for any $\left(\tilde{C}_{1i}^S, \tilde{C}_{2i}^S\right), (\tilde{x}_i, \tilde{y}_i) \in \mathcal{C}_i(A)$.

From Jensen's inequality we obtain that

$$[(1-q)\ln R] - \left[\frac{1}{2}\left((1-p_H)\ln R\frac{(1-q)}{1-p_H} + p_H \ln\frac{q}{p_H}\right) + \frac{1}{2}\left((1-p_L)\ln R\frac{(1-q)}{1-p_L} + p_L \ln\frac{q}{p_L}\right)\right] > 0$$

which implies, by the completeness of real numbers, that there exists a small number $\varphi' > 0$

$$[q\ln 1 + (1-q)\ln R] - \left[\frac{1}{2}\left((1-p_H)\ln R\frac{(1-q)}{1-p_H} + p_H \ln\frac{q}{p_H}\right) + \frac{1}{2}\left((1-p_L)\ln R\frac{(1-q)}{1-p_L} + p_L \ln\frac{q}{p_L}\right)\right] \geq \varphi'.$$

In other words, there exists a small (but independent of $\left(\tilde{C}_{1i}^S, \tilde{C}_{2i}^S\right)$) number $\varphi' > 0$ such that

$$[qu(C_1^*) + (1-q)\,u(C_2^*)] - \left[\frac{1}{2}\left(p_H u(\tilde{C}_{1i}^{S^1}) + (1-p_H)\,u(\tilde{C}_{2i}^{S^1})\right) + \frac{1}{2}\left(p_L u(\tilde{C}_{1i}^{S^2}) + (1-p_L)\,u(\tilde{C}_{2i}^{S^2})\right)\right] \geq \varphi', \tag{40}$$

for any $\left(\tilde{C}_{1i}^S, \tilde{C}_{2i}^S\right), (\tilde{x}_i, \tilde{y}_i) \in \mathcal{C}_A$, where I used that $C_1^* = 1$ and $C_2^* = R$.

Case 1: If $u_i^g \geq u_i^{\tilde{g}}$ it follows immediately that

$$\pi_i\left(g; (C_1^*, C_2^*), (x^*, y^*)\right) > \pi_i\left(\tilde{g}; \left(\tilde{C}_{1i}^S, \tilde{C}_{2i}^S\right), (\tilde{x}_i, \tilde{y}_i)\right)$$

for any $\varphi \in (0,1)$.

Case 2: Suppose that $u_i^g < u_i^{\tilde{g}}$. Let

$$E = \left\{\Delta > 0 \middle|\, \Delta = u_i^{\tilde{g}} - u_i^g\right\}.$$

As consumption is finite, there exists

$$e \equiv \sup(E) \in (0, \infty).$$

Let

$$\bar{\varphi} = \frac{\varphi'}{\varphi' + e}$$

As $\varphi' > 0$ and $e < \infty$, then $\bar{\varphi} > 0$. Then, for any $\varphi \leq \bar{\varphi}$

$$\frac{\varphi}{1-\varphi} e \leq \frac{\bar{\varphi}}{1-\bar{\varphi}} e$$

or

$$\frac{\varphi}{1-\varphi} \left(u_i^{\tilde{g}} - u_i^{g} \right) \leq \varphi'.$$

From (40), we obtain that

$$[qu(C_1^*) + (1-q)\, u(C_2^*)] - \left[\frac{1}{2} \left(p_H u(\tilde{C}_{1i}^{S^1}) + (1-p_H)\, u(\tilde{C}_{2i}^{S^1}) \right) + \frac{1}{2} \left(p_L u(\tilde{C}_{1i}^{S^2}) + (1-p_L)\, u(\tilde{C}_{2i}^{S^2}) \right) \right] \geq \frac{\varphi}{1-\varphi} \left(u_i^{\tilde{g}} - u_i^{g} \right)$$

or

$$\pi_i \left(g; \left(C_1^*, C_2^* \right), \left(x^*, y^* \right) \right) > \pi_i \left(\tilde{g}; \left(\tilde{C}_{1i}^S, \tilde{C}_{2i}^S \right), \left(\tilde{x}_i, \tilde{y}_i \right) \right)$$

for any $\varphi \leq \bar{\varphi}$. Q.E.D.

■

Proof of Proposition 6.

Suppose that $r < \frac{z^*(n+\bar{\eta})}{n((2z+1)(n+\bar{\eta})-1)}$. In the model with linking costs, each bank needs to pay cost ε for each link. Thus, the total cost in a network g_{n,η_i}, where each bank has η_i links with banks in the same region, is $\sum_{i=1}^{2n} (\eta_i + n) \times \varepsilon$. Then, based on the derivations of Proposition 4, the expected welfare in a network g_{n,η_i} in which there are h banks that have $\bar{\eta}$ links and $(2n-h)$ banks that have less then $\bar{\eta}$ links is given by

$$\begin{aligned} W(g_{n,\eta_i}) &= (1-\varphi) \left\{ 2n \times \left[qu\left(C_1^*\right) + (1-q)\, u\left(C_2^*\right) \right] \right\} \\ &\quad + \varphi \left\{ \frac{2n-h}{2n} \left[2n \times u \left(C_1^* - (1-q) \left(C_1^* - \frac{r}{R} C_2^* \right) \right) \right] \right. \\ &\quad + \frac{h}{2n} \left[(2n - (n+\bar{\eta}+1)) \times (qu(C_1^*) + (1-q)\, u\left(C_2^*\right)) + u \left(C_1^* - \frac{(1-q) \left(C_1^* - \frac{r}{R} C_2^* \right)}{1 + (n+\bar{\eta}) \frac{z^*}{n}} \right) \right. \\ &\quad \left. \left. + (n+\bar{\eta}) \times \left(qu(C_1^*) + (1-q)\, u \left(C_2^* - \frac{z^*}{n} \frac{\frac{R}{r} C_1^* - C_2^*}{1 + (n+\bar{\eta}) \frac{z^*}{n}} \right) \right) \right] \right\} \\ &\quad - \sum_{i=1}^{2n} (\eta_i + n) \times \varepsilon. \end{aligned}$$

As shown in the proof of Proposition 4, when the number of banks that have $\bar{\eta}$ links increases from h to $(h+1)$, there is an increases in the aggregate expected utility of consumers of

$$\varphi\left\{-u\left(C_1^*-(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)\right)\right.$$
$$+\frac{1}{2n}\left[(2n-(n+\bar{\eta}+1))\times\left(qu(C_1^*)+(1-q)\,u\left(C_2^*\right)\right)+u\left(C_1^*-\frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+\bar{\eta})\,\frac{z^*}{n}}\right)+\right.$$
$$\left.\left.+\,(n+\bar{\eta})\times\left(qu(C_1^*)+(1-q)\,u\left(C_2^*-\frac{z^*}{n}\frac{\frac{R}{r}C_1^*-C_2^*}{1+(n+\bar{\eta})\frac{z^*}{n}}\right)\right)\right]\right\}.$$

At the same time, in the model with linking costs, the aggregate linking cost increases as well. In particular, the maximum cost that can be associated with an equilibrium interbank network in which there are $(h+1)$ banks with $\bar{\eta}$ links is

$$(h+1)\times\bar{\eta}\times 2\varepsilon+\frac{n\times n}{2}\times 2\varepsilon.$$

This is because in such an interbank network the number of links is maximized when each bank in $H(g_{n,\eta_i})$ has links only with banks in $\{A\cup B\}\setminus H(g_{n,\eta_i})$. Moreover, in an equilibrium interbank network, banks that have less than $\bar{\eta}$ links have no links among themselves (banks in $\{A\cup B\}\setminus H(g_{n,\eta_i})$ have no links with other banks in $\{A\cup B\}\setminus H(g_{n,\eta_i})$).

In addition the minimum cost that can be associated with an equilibrium interbank network in which there are h banks with $\bar{\eta}$ links is

$$\frac{h\times\bar{\eta}}{2}\times 2\varepsilon+\frac{n\times n}{2}\times 2\varepsilon.$$

This is because in such an interbank network the number of links is minimized when each bank in $H(g_{n,\eta_i})$ has links only with banks in $H(g_{n,\eta_i})$. Moreover, in an equilibrium interbank network, banks that have less than $\bar{\eta}$ links have no links among themselves.

This implies that as the number of banks that have $\bar{\eta}$ links increases from h to $(h+1)$, the aggregate linking cost can increase by at most

$$(h+2)\times\bar{\eta}\times\varepsilon\leq(2n+2)\times\bar{\eta}\times\varepsilon.$$

It follows that the expected welfare in an equilibrium interbank network is increasing in h when

$$\begin{aligned}&\varphi\left\{-u\left(C_1^*-(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)\right)\right. \qquad (41)\\&+\frac{1}{2n}\left[(2n-(n+\bar{\eta}+1))\times\left(qu(C_1^*)+(1-q)\,u\left(C_2^*\right)\right)+u\left(C_1^*-\frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+\bar{\eta})\frac{z^*}{n}}\right)+\right.\\&\left.\left.+(n+\bar{\eta})\times\left(qu(C_1^*)+(1-q)\,u\left(C_2^*-\frac{z^*}{n}\frac{\frac{R}{r}C_1^*-C_2^*}{1+(n+\bar{\eta})\frac{z^*}{n}}\right)\right)\right]\right\}\geq(2n+2)\times\bar{\eta}\times\varepsilon.\end{aligned}$$

A sufficient condition that insures that the welfare is increasing is given by

$$\varphi\left\{u\left(C_1^*-\frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+\bar{\eta})\frac{z^*}{n}}\right)-u\left(C_1^*-(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)\right)\right\}>3\varepsilon,$$

or

$$\varphi\,(1-q)\ln\frac{\left(R-\frac{z^*}{n}\frac{\frac{R}{r}-R}{1+(n+\bar{\eta})\frac{z^*}{n}}\right)}{(1-(1-q)\,(1-r))}>3\varepsilon, \qquad (42)$$

if investors have log preferences. If inequality (42) holds, then

$$\varphi\times(2n-(n+\bar{\eta}+1))\times((1-q)\ln R-\ln\left(1-(1-q)\,(1-r)\right))\geq(2n-(n+\bar{\eta}+1))\times\varepsilon,$$

and

$$\varphi\times(n+\bar{\eta})\times\left((1-q)\ln\left(R-\frac{z^*}{n}\frac{\frac{R}{r}-R}{1+(n+\bar{\eta})\frac{z^*}{n}}\right)-\ln\left(1-(1-q)\,(1-r)\right)\right)\geq(n+\bar{\eta})\times\varepsilon,$$

which insures that inequality (41) holds. By defining

$$\bar{\varepsilon}\equiv\frac{1}{3}\varphi\,(1-q)\ln\frac{\left(R-\frac{z^*}{n}\frac{\frac{R}{r}-R}{1+(n+\bar{\eta})\frac{z^*}{n}}\right)}{(1-(1-q)\,(1-r))},$$

the proof is complete. ■

Appendix B: Proof of Proposition 2

A.1 ***Property 1.***

Case 1: $\eta_i \geq \bar{\eta}$ *and* $\eta_j \geq \bar{\eta}$

In this case, the marginal benefit that bank i receives from a link with bank j is given by

$$\pi_i(g_{n,\eta_i}+ij)-\pi_i(g_{n,\eta_i})=(1-q)\,u\left(C_{2i}^d(\eta_j+1)\right)+u\left(C_{1i}^d(\eta_i+1)\right)-(1-q)\,u\left(C_2^*\right)-\left(C_{1i}^d(\eta_i)\right).$$

Thus, Property 1 holds if

$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i+1)\frac{z^*}{n}}\right) - u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i)\frac{z^*}{n}}\right) \leq$$
$$(1-q)\,u\left(C_2^*\right) - (1-q)\,u\left(C_2^* - \frac{z^*}{n}\frac{\frac{R}{r}C_1^* - C_2^*}{1+(n+\eta_j+1)\frac{z^*}{n}}\right).$$

I establish this in three steps.

- **Step 1:** I show first that

$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i+1)\frac{z^*}{n}}\right) - u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i)\frac{z^*}{n}}\right) \leq$$
$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+1)\frac{z^*}{n}}\right) - u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+n\frac{z^*}{n}}\right).$$

  As

$$\frac{1}{1+(n+\eta_i)\frac{z^*}{n}} - \frac{1}{1+(n+\eta_i+1)\frac{z^*}{n}} \leq$$
$$\frac{1}{1+(n+\eta_i-1)\frac{z^*}{n}} - \frac{1}{1+(n+\eta_i)\frac{z^*}{n}}.$$

  then

$$\frac{1}{2}\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i+1)\frac{z^*}{n}}\right) + \frac{1}{2}\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i-1)\frac{z^*}{n}}\right) \leq$$
$$C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i)\frac{z^*}{n}}.$$

  and since $u(\cdot)$ is increasing, then

$$u\left(\frac{1}{2}\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i+1)\frac{z^*}{n}}\right) + \frac{1}{2}\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i-1)\frac{z^*}{n}}\right)\right) \leq$$
$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i)\frac{z^*}{n}}\right).$$

  Using that $u(\cdot)$ is concave, and

$$\frac{1}{2}u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i+1)\frac{z^*}{n}}\right) + \frac{1}{2}u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i-1)\frac{z^*}{n}}\right) \leq$$
$$u\left(\frac{1}{2}\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i+1)\frac{z^*}{n}}\right) + \frac{1}{2}\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i-1)\frac{z^*}{n}}\right)\right).$$

we obtain that

$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i+1)\frac{z^*}{n}}\right) - u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i)\frac{z^*}{n}}\right) \leq$$
$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i)\frac{z^*}{n}}\right) - u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i-1)\frac{z^*}{n}}\right),$$

for any η_i. This implies that

$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i+1)\frac{z^*}{n}}\right) - u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i)\frac{z^*}{n}}\right) \leq$$
$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+1)\frac{z^*}{n}}\right) - u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+n\frac{z^*}{n}}\right).$$

- **Step 2:** As $u(\cdot)$ is increasing, it follows that

$$u(C_2^*) - u\left(C_2^* - \frac{z^*}{n}\frac{\frac{R}{r}C_1^* - C_2^*}{1+(n+\eta_j+1)\frac{z^*}{n}}\right) \geq u(C_2^*) - u\left(C_2^* - \frac{z^*}{n}\frac{\frac{R}{r}C_1^* - C_2^*}{1+(n+n)\frac{z^*}{n}}\right).$$

- **Step 3:** I show that in the case of log-utility function, the following inequality holds

$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+1)\frac{z^*}{n}}\right) - u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+n\frac{z^*}{n}}\right) \leq$$
$$(1-q)\,u(C_2^*) - (1-q)\,u\left(C_2^* - \frac{z^*}{n}\frac{\frac{R}{r}C_1^* - C_2^*}{1+(n+n)\frac{z^*}{n}}\right).$$

In the case of a log-utility function, the optimal solution is given by

$$C_1^* = 1 \text{ and } C_2^* = R$$

As

$$\ln\frac{1 - \frac{\left(1-\frac{r}{R}R\right)}{1+(n+1)\frac{z^*}{n}}}{1 - \frac{\left(1-\frac{r}{R}R\right)}{1+n\frac{z^*}{n}}} \leq \ln\frac{R}{R - \frac{z^*}{n}\frac{\frac{R}{r}-R}{1+(n+n)\frac{z^*}{n}}}$$

and since

$$\frac{1}{1-q}\ln\frac{1 - \frac{(1-q)(1-r)}{1+(n+1)\frac{z^*}{n}}}{1 - \frac{(1-q)(1-r)}{1+n\frac{z^*}{n}}}$$

is decreasing[11] in q, then

$$\frac{1}{1-q}\ln\frac{C_1^*-\frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+1)\frac{z^*}{n}}}{C_1^*-\frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+n\frac{z^*}{n}}}\leq\ln\frac{C_2^*}{C_2^*-\frac{z^*}{n}\frac{\frac{R}{r}C_1^*-C_2^*}{1+(n+n)\frac{z^*}{n}}}.$$

Q.E.D.

Case 2: $\eta_i \leq \bar{\eta}-2$ *and* $\eta_j \geq \bar{\eta}$

In this case, the marginal benefit that bank i receives from a link with bank j is given by

$$\pi_i(g_{n,\eta_i}+ij)-\pi_i(g_{n,\eta_i})=(1-q)\,u\left(C_{2i}^d(\eta_j+1)\right)+u\left(C_1^d\right)-(1-q)\,u\left(C_2^*\right)-u\left(C_1^d\right).$$

Property 1 holds if

$$u\left(C_2^*\right)>u\left(C_2^*-\frac{z^*}{n}\frac{\frac{R}{r}C_1^*-C_2^*}{1+(n+\eta_j+1)\frac{z^*}{n}}\right)$$

which is always true since $u(\cdot)$ is increasing.

Property 2.

Case 1: $\eta_i \geq \bar{\eta}+1$ *and* $\eta_j \geq \bar{\eta}+1$

In this case, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i}-ij)-\pi_i(g_{n,\eta_i})=(1-q)\,u\left(C_2^*\right)+u\left(C_{1i}^d(\eta_i-1)\right)-(1-q)\,u\left(C_{2i}^d(\eta_j)\right)-\left(C_{1i}^d(\eta_i)\right).$$

Thus, Property 2 holds if

$$u\left(C_1^*-\frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+\eta_i)\frac{z^*}{n}}\right)-u\left(C_1^*-\frac{(1-q)\left(C_1^*-\frac{r}{R}C_2^*\right)}{1+(n+\eta_i-1)\frac{z^*}{n}}\right)\leq$$
$$(1-q)\,u\left(C_2^*\right)-(1-q)\,u\left(C_2^*-\frac{z^*}{n}\frac{\frac{R}{r}C_1^*-C_2^*}{1+(n+\eta_j)\frac{z^*}{n}}\right).$$

The derivations are identical as for Property 1.

Case 2: $\eta_i \leq \bar{\eta}-1$ *and* $\eta_j \geq \bar{\eta}+1$

[11] The derivative with respect to q is $\frac{\left((q+z+(1-q)r)(z+nr+nz+nq(1-r))\ln\frac{1-\frac{(1-q)(1-r)}{1+(n+1)\frac{z}{n}}}{1-\frac{(1-q)(1-r)}{1+n\frac{z}{n}}}-z(1-r)(1-q)\right)}{(q-1)^2(q+z+(1-q)r)(z+nr+nz+(1-r)nq)} < -\frac{nz(1-r)^2}{(n+z+nz)(q+r+z-qr)(z+nq+nr+nz-nqr)}$, where I used that $\ln x<(x-1)$ for any $x>0$.

In this case, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i} - ij) - \pi_i(g_{n,\eta_i}) = (1-q)\, u\left(C_2^*\right) + u\left(C_1^d\right) - (1-q)\, u\left(C_{2i}^d(\eta_j)\right) - u\left(C_1^d\right).$$

Property 2 holds if

$$u\left(C_2^*\right) > u\left(C_2^* - \frac{z^*}{n} \frac{\frac{R}{r} C_1^* - C_2^*}{1 + (n+\eta_j)\frac{z^*}{n}}\right)$$

which is always true since $u(\cdot)$ is increasing.

Property 3.

Case 1: $\eta_i \geq \bar{\eta} - 1$ and $\eta_j \leq \bar{\eta} - 1$

When $\eta_i = \bar{\eta} - 1$ and $\eta_j = \bar{\eta} - 1$, the marginal benefit that bank i receives from a link with bank j is given by

$$\pi_i(g_{n,\eta_i} + ij) - \pi_i(g_{n,\eta_i}) = u\left(C_{1i}^d(\eta_i + 1)\right) + u\left(C_{2i}^d\left(\eta_j + 1\right)\right) - u\left(C_1^d\right) - u\left(C_1^d\right).$$

When $\eta_i = \bar{\eta} - 1$ and $\eta_j \leq \bar{\eta} - 2$, the marginal benefit that bank i receives from a link with bank j is given by

$$\pi_i(g_{n,\eta_i} + ij) - \pi_i(g_{n,\eta_i}) = u\left(C_{1i}^d(\eta_i + 1)\right) + u\left(C_1^d\right) - u\left(C_1^d\right) - u\left(C_1^d\right).$$

When $\eta_i \geq \bar{\eta}$ and $\eta_j = \bar{\eta} - 1$, the marginal benefit that bank i receives from a link with bank j is given by

$$\pi_i(g_{n,\eta_i} + ij) - \pi_i(g_{n,\eta_i}) = u\left(C_{1i}^d(\eta_i + 1)\right) + u\left(C_{2i}^d\left(\eta_j + 1\right)\right) - u\left(C_{1i}^d\left(\eta_i\right)\right) - u\left(C_1^d\right).$$

When $\eta_i \geq \bar{\eta}$ and $\eta_j \leq \bar{\eta} - 2$, the marginal benefit that bank i receives from a link with bank j is given by

$$\pi_i(g_{n,\eta_i} + ij) - \pi_i(g_{n,\eta_i}) = u\left(C_{1i}^d(\eta_i + 1)\right) + u\left(C_1^d\right) - u\left(C_{1i}^d\left(\eta_i\right)\right) - u\left(C_1^d\right).$$

Property 3 holds, as it follows from (12), (11), and (14) that

$$C_{2i}^d\left(\eta_j + 1\right) \geq C_1^* \geq C_{1i}^d(\eta_i + 1) \geq C_{1i}^d(\eta_i) \geq C_1^d.$$

Case 2: $\eta_i \leq n - 1$ and $\eta_j = \bar{\eta} - 1$

When $\eta_i \geq \bar{\eta}$, the marginal benefit that bank i receives from a link with bank j is given by

$$\pi_i(g_{n,\eta_i} + ij) - \pi_i(g_{n,\eta_i}) = u\left(C_{1i}^d(\eta_i + 1)\right) + u\left(C_{2i}^d\left(\eta_j + 1\right)\right) - u\left(C_{1i}^d(\eta_i)\right) - u\left(C_1^d\right).$$

When $\eta_i = \bar{\eta} - 1$, the marginal benefit that bank i receives from a link with bank j is given by

$$\pi_i(g_{n,\eta_i} + ij) - \pi_i(g_{n,\eta_i}) = u\left(C_{1i}^d(\eta_i + 1)\right) + u\left(C_{2i}^d\left(\eta_j + 1\right)\right) - u\left(C_1^d\right) - u\left(C_1^d\right).$$

When $\eta_i \leq \bar{\eta} - 2$, the marginal benefit that bank i receives from a link with bank j is given by

$$\pi_i(g_{n,\eta_i} + ij) - \pi_i(g_{n,\eta_i}) = u\left(C_1^d\right) + u\left(C_{2i}^d\left(\eta_j + 1\right)\right) - u\left(C_1^d\right) - u\left(C_1^d\right).$$

Property 3 holds, as it follows from (12), (11), and (14) that

$$C_{2i}^d\left(\eta_j + 1\right) \geq C_1^* \geq C_{1i}^d(\eta_i + 1) \geq C_{1i}^d(\eta_i) \geq C_1^d.$$

Property 4.

Case 1: $\eta_i \geq \bar{\eta}$ and $\eta_j \leq \bar{\eta}$

When $\eta_i = \bar{\eta}$ and $\eta_j = \bar{\eta}$, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i} - ij) - \pi_i(g_{n,\eta_i}) = u\left(C_1^d\right) + u\left(C_1^d\right) - u\left(C_{1i}^d(\eta_i)\right) - u\left(C_{2i}^d\left(\eta_j\right)\right).$$

When $\eta_i = \bar{\eta}$ and $\eta_j \leq \bar{\eta} - 1$, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i} - ij) - \pi_i(g_{n,\eta_i}) = u\left(C_1^d\right) + u\left(C_1^d\right) - u\left(C_{1i}^d(\eta_i)\right) - u\left(C_1^d\right).$$

When $\eta_i \geq \bar{\eta} + 1$ and $\eta_j = \bar{\eta}$, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i} - ij) - \pi_i(g_{n,\eta_i}) = u\left(C_{1i}^d\left(\eta_i - 1\right)\right) + u\left(C_1^d\right) - u\left(C_{1i}^d(\eta_i)\right) - u\left(C_{2i}^d\left(\eta_j\right)\right).$$

When $\eta_i \geq \bar{\eta}+1$ and $\eta_j \leq \bar{\eta}-1$, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i} - ij) - \pi_i(g_{n,\eta_i}) = u\left(C_{1i}^d(\eta_i - 1)\right) + u\left(C_1^d\right) - u\left(C_{1i}^d(\eta_i)\right) - u\left(C_1^d\right).$$

Property 4 holds, as it follows from (12), (11), and (14) that

$$C_{2i}^d\left(\eta_j + 1\right) \geq C_1^* \geq C_{1i}^d(\eta_i + 1) \geq C_{1i}^d(\eta_i) \geq C_1^d.$$

Case 2: $\eta_i \leq n-1$ and $\eta_j = \bar{\eta}$

When $\eta_i \geq \bar{\eta}+1$, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i} - ij) - \pi_i(g_{n,\eta_i}) = u\left(C_{1i}^d(\eta_i - 1)\right) + u\left(C_1^d\right) - u\left(C_{1i}^d(\eta_i)\right) - u\left(C_{2i}^d\left(\eta_j\right)\right).$$

When $\eta_i = \bar{\eta}$, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i} - ij) - \pi_i(g_{n,\eta_i}) = u\left(C_1^d\right) + u\left(C_1^d\right) - u\left(C_{1i}^d(\eta_i)\right) - u\left(C_{2i}^d\left(\eta_j\right)\right).$$

When $\eta_i \leq \bar{\eta}-1$, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i} - ij) - \pi_i(g_{n,\eta_i}) = u\left(C_1^d\right) + u\left(C_1^d\right) - u\left(C_1^d\right) - u\left(C_{2i}^d\left(\eta_j\right)\right).$$

Property 4 holds, as it follows from (12), (11), and (14) that

$$C_{2i}^d\left(\eta_j + 1\right) \geq C_1^* \geq C_{1i}^d(\eta_i + 1) \geq C_{1i}^d(\eta_i) \geq C_1^d.$$

Property 5.

$\eta_i = \bar{\eta} - 1$ *and* $\eta_j \geq \bar{\eta}$ *if* $r < \frac{z^*(n+\bar{\eta})}{n((2z+1)(n+\bar{\eta})-1)}$.

In this case, the marginal benefit that bank i receives from a link with bank j is given by

$$\pi_i(g_{n,\eta_i} + ij) - \pi_i(g_{n,\eta_i}) = (1-q)\,u\left(C_{2i}^d(\eta_j + 1)\right) + u\left(C_{1i}^d(\eta_i + 1)\right) - (1-q)\,u\left(C_2^*\right) - u\left(C_1^d\right).$$

Property 5 holds if

$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i+1)\frac{z^*}{n}}\right) - u\left(C_1^* - (1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)\right) \leq$$

$$(1-q)\,u\left(C_2^*\right) - (1-q)\,u\left(C_2^* - \frac{z^*}{n}\frac{\frac{R}{r}C_1^* - C_2^*}{1+(n+\eta_j+1)\frac{z^*}{n}}\right)$$

I establish this in three steps.

- **Step 1**: As $u(\cdot)$ is increasing, it follows that

$$u\left(C_2^*\right) - u\left(C_2^* - \frac{z^*}{n}\frac{\frac{R}{r}C_1^* - C_2^*}{1+(n+\eta_j+1)\frac{z^*}{n}}\right) >$$

$$u\left(C_2^*\right) - u\left(C_2^* - \frac{z^*}{n}\frac{\frac{R}{r}C_1^* - C_2^*}{1+2z}\right)$$

- **Step 2**: Next I show that

$$(1-q)\,u\left(C_2^*\right) - (1-q)\,u\left(C_2^* - \frac{z^*}{n}\frac{\frac{R}{r}C_1^* - C_2^*}{1+2z}\right) >$$

$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\bar{\eta})\frac{z^*}{n}}\right) - u\left(C_1^* - (1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)\right)$$

  In the case of a log-utility function, the optimal solution is given by

$$C_1^* = 1 \text{ and } C_2^* = R$$

  As, by assumption,

$$r < \frac{z^*(n+\bar{\eta})}{n\left((2z+1)(n+\bar{\eta})-1\right)},$$

  then

$$\frac{R}{R - \frac{z^*}{n}\frac{\frac{R}{r}-R}{1+2z}} \geq \frac{1 - \frac{(1-r)}{1+(n+\bar{\eta})\frac{z^*}{n}}}{(1-(1-r))}.$$

  Moreover, since

$$\frac{1}{1-q}\ln\frac{1 - \frac{(1-q)(1-r)}{1+(n+\bar{\eta})\frac{z^*}{n}}}{(1-(1-q)(1-r))}$$

  is decreasing[12] in q, then

$$\ln\frac{C_2^*}{C_2^* - \frac{z^*}{n}\frac{\frac{R}{r}C_1^* - C_2^*}{1+2z}} > \frac{1}{1-q}\ln\frac{C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta)\frac{z^*}{n}}}{\left(C_1^* - (1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)\right)}.$$

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[12] The derivative with respect to q is $\frac{(q+(1-q)r)(z(\bar{\eta}+n)+nq+(1-q)nr)\ln\frac{1-\frac{(1-q)(1-r)}{1+(n+\bar{\eta})\frac{z}{n}}}{(1-(1-q)(1-r))} - z(1-r)(1-q)(n+\bar{\eta})}{(1-q)^2(q+(1-q)r)(z(\bar{\eta}+n)+nq+(1-q)nr)} < -\frac{nz(n+\bar{\eta})(1-r)^2}{(n+z(\bar{\eta}+n))(q+(1-q)r)(z(\bar{\eta}+n)+nq+(1-q)nr)}$, where I used that $\ln x < (x-1)$ for any $x > 0$.

Q.E.D.

There is still an issue of the existence of the parameter space. Note that, for a positive $\bar{\eta} \leq n-1$ to exist, inequality (13) implies that

$$\frac{Rz}{z^* + n\left(2z+1\right)\left(R-1\right)} \leq r \leq \frac{Rz}{\left(R-1\right)n\left(1+z^*\right) + Rz}.$$

As $\bar{\eta}$ is the smallest integer that satisfies (13), it follows that if

$$r - \frac{z^*\left(\frac{(1-r)}{R-1}\frac{R}{r} - \frac{n}{z^*} - n + 1 + n\right)}{n\left(\left(2z+1\right)\left(n + \frac{(1-r)}{R-1}\frac{R}{r} - \frac{n}{z^*} - n + 1\right) - 1\right)} < 0, \tag{43}$$

then

$$r < \frac{z^*\left(\bar{\eta}+n\right)}{n\left(\left(2z+1\right)\left(n+\bar{\eta}\right)-1\right)}.$$

The inequality (43) holds for any r such that

$$f(r) = \left(-n^2\left(2z+1\right)\left(R-1\right) - Rnz - 2nz^2\right)r^2 + \left(z^{*2} - nz + 2Rnz^2 + 2Rnz\right)r - Rz^2 < 0.$$

The $f(r)$ is quadratic and concave in r, and it reaches a maximum for

$$r_{\max} = \frac{z^{*2} - nz + 2Rnz^2 + 2Rnz}{n^2\left(2z+1\right)\left(R-1\right) + Rnz + 2nz^2}.$$

It is then sufficient if

$$f\left(\frac{Rz}{z^* + n\left(2z+1\right)\left(R-1\right)}\right) < 0,$$

or, equivalently, if

$$R - 2n + 2z - 4nz > 0,$$

to show that there exist $r > \frac{Rz}{z^*+n(2z+1)(R-1)}$, such that $f(r) < 0$.

Property 6.

$\eta_i = \bar{\eta}$ and $\eta_j \geq \bar{\eta}+1$ if $r < \frac{z^*(n+\bar{\eta})}{n((2z+1)(n+\bar{\eta})-1)}$.

In this case, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i} - ij) - \pi_i(g_{n,\eta_i}) = \left(1-q\right)u\left(C_2^*\right) + u\left(C_1^d\right) - \left(1-q\right)u\left(C_{2i}^d(\eta_j)\right) - u\left(C_{1i}^d(\eta_i)\right).$$

Property 6 holds if

$$u\left(C_1^* - \frac{(1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)}{1+(n+\eta_i)\frac{z^*}{n}}\right) - u\left(C_1^* - (1-q)\left(C_1^* - \frac{r}{R}C_2^*\right)\right) \leq$$
$$(1-q)\,u\left(C_2^*\right) - (1-q)\,u\left(C_2^* - \frac{z^*}{n}\frac{\frac{R}{r}C_1^* - C_2^*}{1+(n+\eta_j)\frac{z^*}{n}}\right)$$

The derivations are identical as for Property 5.

Property 7.

$\eta_i \leq \bar{\eta} - 2$ *and* $\eta_j \leq \bar{\eta} - 2$

In this case, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i} + ij) - \pi_i(g_{n,\eta_i}) = u\left(C_1^d\right) + u\left(C_1^d\right) - u\left(C_1^d\right) - u\left(C_1^d\right),$$

or

$$\pi_i(g_{n,\eta_i} + ij) = \pi_i(g_{n,\eta_i})$$

Property 8.

$\eta_i \leq \bar{\eta} - 1$ *and* $\eta_j \leq \bar{\eta} - 1$

In this case, the marginal benefit that bank i receives from severing a link with bank j is given by

$$\pi_i(g_{n,\eta_i} - ij) - \pi_i(g_{n,\eta_i}) = u\left(C_1^d\right) + u\left(C_1^d\right) - u\left(C_1^d\right) - u\left(C_1^d\right),$$

or

$$\pi_i(g_{n,\eta_i} - ij) = \pi_i(g_{n,\eta_i})$$