

Contagion Risk in Financial Networks

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Abstract

Banking systems can easily be described by a network, where the links take the form of direct exposures between banks. The same connections that facilitate the transfer of liquidity between banks, expose the banking system to the risk of contagion. That is, idiosyncratic shocks, which initially affect only a few institutions, may propagate through the entire system. This paper studies how the trade-off between the benefits and the costs of being linked changes depending on the network structure. We have shown that incomplete networks give rise to incomplete information. In this situation, the transfers between banks that perfectly ensures against liquidity shocks increase, at the same time, the contagion risk. The problem is solved when the network is complete, as the liquidity can be redistributed in the system, such that the risk of contagion is minimal.

Keywords: financial stability; interbank deposits; uncertainty; complete and incomplete networks.

JEL: G21; D82.

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1 Introduction

A notable feature of the modern financial world is its high degree of interdependence. Banks and other financial institutions are linked in a variety of ways. The mutual exposures that financial institutions adopt towards each other connect the banking system in a network. Despite their obvious benefit, the linkages come at the cost that shocks, which initially affect only a few institutions, can propagate through the entire system. Since these linkages carry the risk of contagion, an interesting question is whether the degree of interdependence in the banking system sustains systemic stability. This paper addresses this issue. In particular, we investigate how the network structure affects the cross institutional holdings and investigate the implications for contagion risk.

We study a setting where the banking system is exposed to both liquidity and idiosyncratic shocks. The connections between banks facilitate the transfer of liquidity from the ones that have a cash surplus to those with a cash deficit. Banks can insure against the liquidity shocks by exchanging deposits through links in the network. The same connections, however, make the banking system prone to contagion. The risk of contagion is increasing in the size of interbank deposits, as we show in this paper. We first investigate the size of interbank deposits transferred between banks that provides full insurance against liquidity shocks, while keeping the network structure fixed. Then we assess how the banking system responds to contagion.

We show that incomplete network structures create uncertainty about the distribution of liquidity shocks. As a result, the level of interbank deposits that insures against liquidity shocks increases, at the same time, the contagion risk. In other words, to achieve perfect insurance against liquidity shocks, banks need to accept a higher risk of contagion. The problem is solved when the network is complete, as the level of deposits that perfectly re-distribute the liquidity in the banking system, also minimizes the risk of contagion.

Recently, there has been a substantial interest in looking for evidence of contagious failures of financial institutions resulting from the mutual claims they have on one another. Most of these papers use balance sheet information to estimate bilateral credit relationships for different banking systems. Subsequently, the stability of the interbank market is tested by simulating the breakdown of a single bank. Upper and Worms (2004) analyze the German banking system. Sheldon and Maurer (1998) consider the Swiss system. Cocco et al. (2005) present empirical evidence for lending relationships existent on the portuguese

interbank market. Furfine (2003) studies the interlinkages between the US banks, while Wells (2004) looks at the UK interbank market. Boss et al. (2004) provide an empirical analysis of the network structure of the Austrian interbank market and discuss its stability when a node is eliminated. In the same manner, Degryse and Nguyen (2007) evaluate the risk that a chain reaction of bank failures would occur in the Belgian interbank market. These papers find that the banking systems demonstrate a high resilience, even to large shocks. Simulations of the worst case scenarios show that banks representing less than five percent of total balance sheet assets would be affected by contagion on the Belgian interbank market, while for the German system the failure of a single bank could lead to the breakdown of up to 15% of the banking sector in terms of assets. In this paper, we advance an explanation for this apparent stability of the financial systems, in an attempt to fill the gap between the relatively skeptical theoretical models and the good news brought by the empirical research.

The theoretical papers which study banking contagion paint a more pessimistic message. There are two approaches in this literature. On the one hand, there is a number of papers that look for contagious effects via indirect linkages. Lagunoff and Schreft (2001) construct a model where agents are linked in the sense that the return on an agent's portfolio depends on the portfolio allocations of other agents. Similarly, de Vries (2005) shows that there is dependency between banks' portfolios, given the fat tail property of the underlying assets, and this carries the potential for systemic breakdown. Cifuentes et al. (2005) present a model where financial institutions are connected via portfolio holdings. The network is complete as everyone holds the same asset. Although the authors incorporate in their model direct linkages through mutual credit exposures as well, contagion is mainly driven by changes in asset prices. These papers, they all share the same finding: financial systems are inherently fragile. Fragility, not only arises exogenously, from financial institutions' exposure to macro risk factors, as it is the case in de Vries (2005). It also endogenously evolves through forced sales of assets by some banks that depress the market price inducing further distress to other institutions, as in Cifuentes et al. (2004).

The other approach focuses on direct balance sheet interlinkages. For instance, Freixas et al. (2000) considers the case of banks that face liquidity needs as consumers are uncertain about where they are to consume. In their model the connections between banks are realized through interbank credit lines that enable these institutions to hedge regional

liquidity shocks. The authors analyze different market structures and find that a system of credit lines, while it reduces the cost of holding liquidity, makes the banking sector prone to experience gridlocks, even when all banks are solvent. Dasgupta (2004) also discusses how linkages between banks represented by crossholding of deposits can be a source of contagious breakdowns. Fragility arises when depositors, that receive a private signal about banks' fundamentals, may wish to withdraw their deposits if they believe that enough other depositors will do the same. To eliminate the multiplicity of equilibria Dasgupta (2004) uses the concept of global games. The author isolates a unique equilibrium which depends on the value of the fundamentals. Eisenberg and Noe (2001) take a more technical approach when investigating systemic risk in a network of financial institutions. First the authors show the existence of a clearing payment vector that defines the level of connections between banks. Next, they develop an algorithm that allows them to evaluate the effects small shocks have on the system. Another interesting issue is addressed by Leitner (2005). The model constructed in this paper shows that agents may be willing to bail out other agents, in order to prevent the collapse of the whole network.

The paper that is closest related to ours is by Allen and Gale (2000). They asses the impact of the degree of network completeness on the stability of the banking system. Allen and Gale show that complete networks are more resilient to contagious effects of a single bank failure than incomplete structures. In their model, though there is no aggregate shortage of liquidity, the demand for cash is not evenly distributed in the system. This induces banks to insure against such regional liquidity shocks by exchanging deposits on the interbank market. The interbank market is perceived as a network where the banks are nodes and the deposits exchanged represent links.

Our paper uses the same framework as Allen and Gale (2000) to motivate interactions on the interbank market. However, Allen and Gale (2000) study the banking system when there exist correlations between the shocks in the liquidity demand that affect different regions. We extend their analyses and look at the banking system without building in any correlations between liquidity shocks. Thus, we introduce uncertainty about what regions have negatively correlated shocks. In addition, we incorporate one very important feature of real world banking systems. That is, relations between banks, in general, and deposit contracts, in particular are private information. Our setting captures this aspect and allows a link that exists between two banks not to be observed by the other banks

in the system. Thus, we analyze network effects on systemic risk if these two sources of uncertainty are present.

Allen and Gale (2000) find that the losses caused by contagion in an incomplete network are larger than in a complete network. We reinforce their result by showing that incomplete networks have an additional effect. When the network is incomplete, the allocation of interbank deposits that provides insurance against liquidity shocks is unlikely at the level that minimizes contagion risk. This is no longer the case when the network becomes complete. In a complete network the degree of interdependence between banks is such that contagion risk is minimum.

The model is based on a framework introduced by Diamond and Dybvig (1983). There are three periods $t = 0, 1, 2$ and a large number of identical consumers, each endowed with one unit of a consumption good. Ex-ante, consumers are uncertain about their liquidity preferences. They might be early consumers, who value consumption at date 1, or late consumers, who value consumption at date 2. The consumers find optimal to deposit their endowment in banks, which invest on their behalf. In return, consumers are offered a fixed amount of consumption at each subsequent date, depending when they choose to withdraw. Banks can invest in two assets: there is a liquid asset which pays a return of 1 after one period and there is an illiquid asset that pays a return of $r < 1$ after one period or $R > 1$ after two periods. In addition, liquidity shocks hit the economy randomly, in the following way. Although there is no uncertainty about the average fraction of early consumers, the liquidity demand is unevenly distributed among banks in the first period. Thus, each bank experiences either a high or a low fraction of early consumers. To ensure against these regional liquidity shocks, banks exchange deposits on the interbank market in period 0.

Deposits exchanged this way constitute the links that connect the banks in a network. This view of the banking system as a network is useful in analyzing the effects that the failure of a bank may produce. If such an event occurs, the risk of contagion is evaluated in terms of the loss in value for the deposits exchanged at date 0. It becomes apparent that contagion risk depends on the size of these deposits. When the probability of a bank failure is small, the size of the interbank deposits needs to meet two criteria. First, interbank deposits should be large enough to insure perfectly against any distribution of liquidity shocks realized at date 1. And second, interbank deposits should not be larger

than needed for insurance against liquidity shocks. In other words, they need to minimize the risk of contagion by diversification.

The paper is organized as follows. Section 2 introduces the main assumptions about consumers and banks and describes the interbank market as a network. We discuss the linkages between banks and how contagion may arise in section 3. In Section 4 we show how banks set the interbank deposits and investigate if they are at the level that minimizes contagion risk for different degrees of network connectedness. Section 5 considers possible extensions and ends with some concluding remarks.

2 The Model

2.1 Consumers and Liquidity Shocks

We assume that the economy is divided into 6 regions, each populated by a continuum of risk averse consumers (the reason for 6 will become clear in due course). There are three time periods $t = 0, 1, 2$. Each agent has an endowment equal to one unit of consumption good at date $t = 0$. Agents are uncertain about their liquidity preferences: they are either early consumers, who value consumption only at date 1, or they are late consumers, who value consumption only at date 2. In the aggregate there is no uncertainty about the liquidity demand in period 1. Each region, however, experiences different liquidity shocks, caused by random fluctuations in the fraction of early consumers. In other words, each region will face either a high proportion p_H of agents that need to consume at date 1 or a low proportion p_L of agents that value consumption in period 1. There are $\binom{6}{3}$ equally likely states of nature that distribute the high liquidity shocks to exactly three regions and the low liquidity shocks to the other three. One may note that this set of states of the world does not build in any correlations between the liquidity shocks that affect any two regions.

To sum up, it is known with certainty that on average the fraction of early consumers in the economy is $q = (p_H + p_L)/2$. Nevertheless, the liquidity demand is not uniformly distributed among regions. All the uncertainty is resolved at date 1, when the state of the world is realized and commonly known. At date 2, the fraction of late consumers in each region will be $(1 - p)$ where the value of p is known at date 1 as either p_H or p_L .

2.2 Banks, Demand Deposits and Asset Investments

We consider that in each region i there is a competitive representative bank. Agents deposit their endowment in the regional bank. In exchange, they receive a deposit contract that guarantees them an amount of consumption depending on the date they choose to withdraw their deposits. In particular, the deposit contract specifies that if they withdraw at date 1, they receive $C_1 > 1$, and if they withdraw at date 2, they receive $C_2 > C_1$.

There are two possibilities to invest. First, banks can invest in a liquid asset with a return of 1 after one period. They can also choose an illiquid asset that pays a return of $r < 1$ after one period, or $R > 1$ after two periods. Let x and y be the per capita amounts invested in the liquid and illiquid asset, respectively. Banks will use the liquid asset to pay depositors that need to withdraw in the first period and will reserve the illiquid asset to pay the late consumers. Since the average level of liquidity demand at date 1 is qC_1 , we assume that the investment in the liquid asset, x , will equal this amount, while the investment in the illiquid asset, y , will cover $(1 - q)C_2/R$.¹ This macro allocation will be relaxed later.

Banks are subject to idiosyncratic shocks that are not insurable. That means that, with a small probability π , the failure of a bank will occur in either period 1 or 2. This event, although anticipated, will have only a secondary effect on banks' actions for reasons that will become clear in section 4.

2.3 Interbank Market

Uncertainty in their depositors' preferences motivates banks to interact in order to ensure against the liquidity shocks that affect the economy. These interactions create balance sheet linkages between banks, as described below.

At date 1 each bank has with probability half either a liquidity shortage of $(p_H - q)C_1$ or a liquidity surplus of $(q - p_L)C_1$. We denote by z the deviation from the mean of the fraction of early consumers, which in turn makes the liquidity surplus or shortage of a banks equal to zC_1 .² As in the aggregate, the liquidity demand matches the liquidity supply, all the regional imbalances can be solved by the transfer of funds from banks with a cash surplus to banks with a cash deficit. Anticipating this outcome, banks will

¹This allocation maximizes the expected utility of consumers, see Allen and Gale (2000).

²Since $q = \frac{p_H + p_L}{2}$, than it must be that $(p_H - q)C_1 = (q - p_L)C_1$.

agree to hedge the regional liquidity shocks by exchanging deposits at date 0. This way, a contract is closed between two banks that gives the right to both parts to withdraw their deposit, fully or only in part, at any of the subsequent dates. For the amounts exchanged as deposits, each bank receives the same return as consumers: C_1 , if they withdraw after one period, and C_2 if they withdraw after two periods.

Banks' portfolios consist now of three assets: the liquid asset, the illiquid asset and the interbank deposits. Each of these three assets can be liquidated in any of the last 2 periods. However, the costliest in terms of early liquidation is the illiquid asset. This implies the following ordering of returns:

$$1 < \frac{C_2}{C_1} < \frac{R}{r} \quad (2.1)$$

An important feature of the model is that the swap of deposits occurs ex-ante, before the state of the world is realized. Note, however, that this prevents cases when lenders have some monopoly power to arise. For instance, in an ex-post market for deposits, lenders might take advantage of their position as liquidity providers to extract money from banks with a shortage of liquidity. To avoid this unfavorable situation, banks prefer to close firm contracts that set the price of liquidity ex-ante.

An interbank market, as introduced above may be very well described as a network. The network can be characterized by the pattern of interactions between banks, as well as by the amount of interbank deposits that represent the links. In this paper, we investigate how the size of interbank deposits, depends on the network structure. In particular, we are interested in the effects complete and incomplete networks have on banks' decisions when setting the level of interbank deposits. In order to illustrate the effects of incomplete structures, we restrict our analysis to regular networks (we introduce definitions below). Thus, each bank in the network is a node and each node is connected to exactly $n < 6$ other nodes. This means that each bank may, but need not, exchange deposits with other n banks. Note that we do not model explicitly how these connections are formed. Since the contracts are bilateral, and thus the amounts exchanged between any two banks are the same, the network is *undirected*. Next, we introduce a some important definitions.

A *network* g is, formally, a collection of ij pairs, with the interpretation that nodes i and j are linked. A network is *regular of degree* n (or *n -regular*) if any node in the network

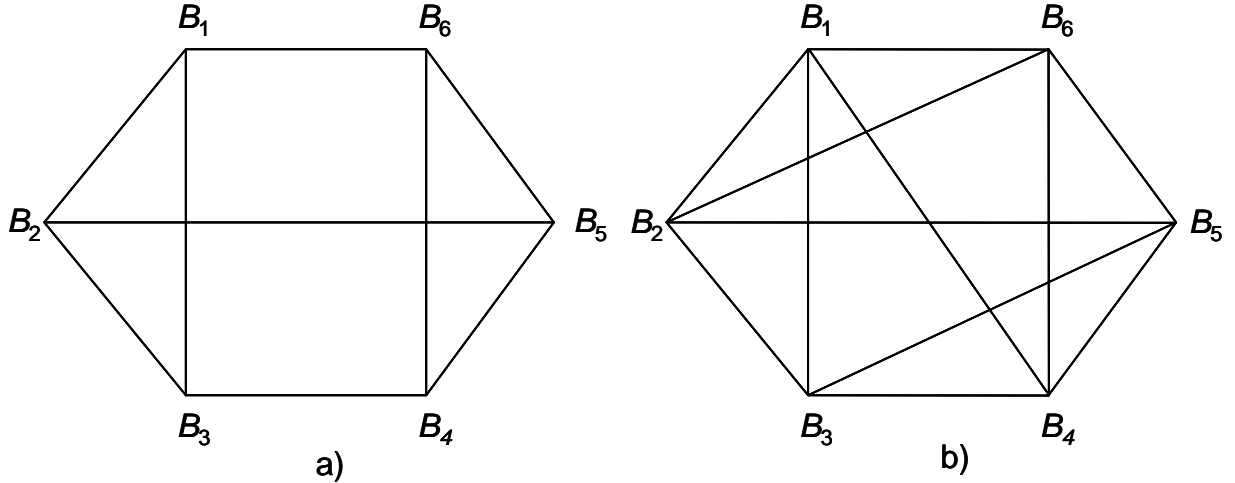


Figure 2.1: n -regular networks: a) $n = 3$; b) $n = 4$

is directly connected with other n nodes. The *complete network* is the graph in which all nodes are linked to one another. Any two nodes connected by a link are called *neighbors*.

We now discuss the incomplete information structure. We incorporate in our framework an important feature of real world banking systems. Namely, banks have incomplete information over the network structure. Although it is common knowledge that the network is n -regular, banks do not know the entire network architecture. Thus, they do not observe the linkages in the network, beyond their own connections. For instance, B_1 in figure 2.1 knows his set of neighbors: B_2 , B_3 and B_6 . Nevertheless, it cannot observe how they are connected neither between themselves, nor to the other banks in the system.

For the purposes of our analyses we consider different values of n . However, since modern banking systems are highly connected, we reasonably assume that $n \geq 3$.³ In other words, each bank is connected to at least half of the other banks in the system. At the same time the markets are not always complete structures. In a possible interpretation, in a single country interbank market all the banks are connected to all the other banks. The connections outside the home country are nevertheless rather scarce.

³The cases $n = 1$ and $n = 2$ will be shortly discuss later in the paper.

3 Contagion Risk

3.1 Balance Sheet Linkages

The main goal of our paper is to study which degree of interdependence arises between banks and the implications for the fragility of the banking system. The interdependence stems from two sources. First, there is a system-wide dependence that is reflected in the size of z , the liquidity shortage or surplus of any bank. The larger z , the higher is the degree of interdependence. Second, there is pairwise dependence that is given by the size of deposits exchanged between any two banks. Since we assume z to be fixed, for the moment, we focus on explaining pairwise dependence and its potential contagious consequences.

An *allocation rule for deposits* is a mapping from the set of links to the real numbers $a : g \rightarrow \mathbb{R}$ that specifies the amount exchanged as deposits between banks i and j at date 0. For simplicity we use the following notation $a(ij) = a_{ij}$. As in the previous section we considered that deposit contracts are bilateral, we have $a_{ij} = a_{ji}$, thus bilateral interbank deposits.

We consider that an allocation rule is *feasible* if in period 1 deposits can be withdrawn such that there will be no bank with a liquidity surplus nor a liquidity shortage. Formally, let d_{ij} represent the amount transferred from i to j in period 1, for any pair ij , and N_i be the set of neighbors of bank i , for any i . Then, an allocation rule is feasible if, for any bank i and for any neighbor j of i , there exist d_{ij} and d_{ji} such that $\left| \sum_{j \in N_i} d_{ji} - \sum_{j \in N_i} d_{ij} \right| C_1 = zC_1$ and $0 \leq d_{ij}, d_{ji} \leq a_{ij}$.⁴

Lemma 1 *For a n -regular network with $n \geq 3$ there always exists a feasible allocation rule.*

Proof. This holds true as in a n -regular network, when $n \geq 3$, there is always a path between every pair of nodes. A path is a sequence of consecutive links in a network.

⁴Note that $d_{ij} \neq d_{ji}$ in period 1. That is because when the state of the world is realized in period 1, liquidity will flow from banks that have in excess to banks that have a deficit. Hence, the network becomes directed in period 1.

Moreover, it can be shown that the length of this path it is at most 2. A general proof follows in the appendix. ■

The proof of Lemma 1 shows in fact that there exists a feasible allocation for any connected network. A regular network with a degree larger than half the number of nodes is a particular case of connected network.

Corollary 1 *A feasible allocation ensures that there will be no bank with a liquidity surplus nor a liquidity shortage in period 2 as well.*

In period 2 each bank will have a fraction of $(1 - p)$ late consumers where p has been realized for each region in period 1. Thus, the transfer of deposits between any banks i and j will simply be reversed.

3.2 Losses Given Default

In order to evaluate contagion risk we need to introduce a measure that quantifies it. For this purpose, we apply the same procedure as the empirical literature on contagion: we consider the event of a bank failure and analyze its implications for the banking system. In our model, the failure of a bank will occur in either period 1 or 2 with a small probability π . The risk of contagion is then evaluated in terms of *loss given default* (henceforth LGD). LGD expresses the excess of nominal liabilities over the value of the assets of the failed bank. In our setting, LGD will be given by the loss of value a bank incurs on its deposits when one of its neighbor banks is liquidated.

This measure focuses only on the loss associated to a direct link between two banks. It ignores any aspects related to the indirect effects the failure of a bank might have on the system. For instance, it does not capture the problems that arise when a bank that is a liquidity supplier fails.

Another aspect worth mentioning is that the failure of a bank might have contagious effects only if this event is realized in period 1. Once each bank reaches period 2, straightforward calculations show that the value of its assets is sufficiently large to cover all its liabilities. Hence, there is no loss in value for deposits, and LGD will be 0.

To calculate LGD we need to determine the value of the assets of the failed bank. If a bank fails, its portfolio of assets is liquidated at the current value and distributed equally among creditors. Now, recall that a bank portfolio consists of three assets. First,

banks hold an amount of x per capita invested in a liquid asset that pays a return of 1. Second, banks have invested an amount y per capita in an illiquid asset that pays a return of $r < 1$ if liquidated in the first period. And lastly, there are interbank deposits summing up to $\sum_{k \in N_i} a_{ik}$ that pay a return of C_1 per unit of deposit. On the liability side, a bank will have to pay its depositors, normalized to 1 and at the same time to repay its interbank creditors that also add up to $\sum_{k \in N_i} a_{ik}$. This yields a new return per unit of good deposited in a bank i equal to $\bar{C}_i = \frac{x+ry+\sum_{k \in N_i} a_{ik}C_1}{1+\sum_{k \in N_i} a_{ik}} < C_1$.⁵ The LGD of bank j given that bank i has failed is easy now to express as⁶:

$$LGD_{ji} = a_{ji}(C_1 - \bar{C}_i) = a_{ji} \frac{C_1 - x - ry}{1 + \sum_{k \in N_i} a_{ik}} \quad (3.1)$$

4 Deposits Allocation and their Optimality

4.1 Network Structures and Uncertainty

To understand how the allocation of deposits should be set in period 0, we need to characterize the sources of uncertainty that dominate the environment in the banking system.

In an incomplete network, there are two sources of uncertainty. On the one hand, there is no prior information about the distribution of liquidity shocks. That is, any of the $\binom{6}{3}$ states of the world that allows a high liquidity demand in any 3 regions and a low liquidity demand in the remaining 3 is equally likely. This further implies that there is no ex-ante correlation between the fractions of early consumers in any two regions. The lack of correlations between liquidity shocks is converted, for any bank i , into uncertainty. First, there is uncertainty about how many neighbors from N_i will be affected by a different liquidity shock than i at date 1. And second, there is uncertainty about who these neighbors are. Note that the first type of uncertainty depends on the network degree of completeness n and disappears when the network is complete. That is because the condition $n \geq 3$ guarantees that each bank has at least $n - 2$ neighbors that will face a different liquidity demand in period 1.

Example 1 *Suppose that the network degree is $n = 3$. Then a bank might have, as regarded from period 0, one, two or three neighbors that may experience a different fraction*

⁵Eq. (2.1) ensures that the inequality holds.

⁶In principle $LGD_{ji} \neq LGD_{ij}$ since it may be that $\sum_{k \in N_i} a_{ik} \neq \sum_{k \in N_k} a_{jk}$

of early consumers than itself in period 1.

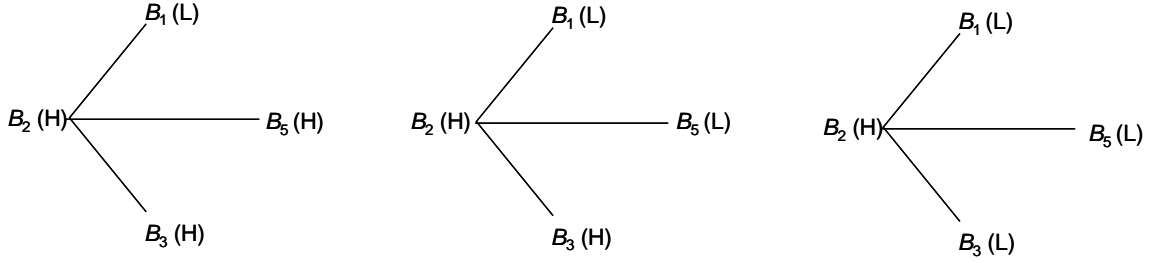


Figure 4.1: Uncertainty about the number of neighbours of a different type

Moreover, any of the banks in the neighbors set of a bank i , is equally likely to experience a different liquidity shock than i .

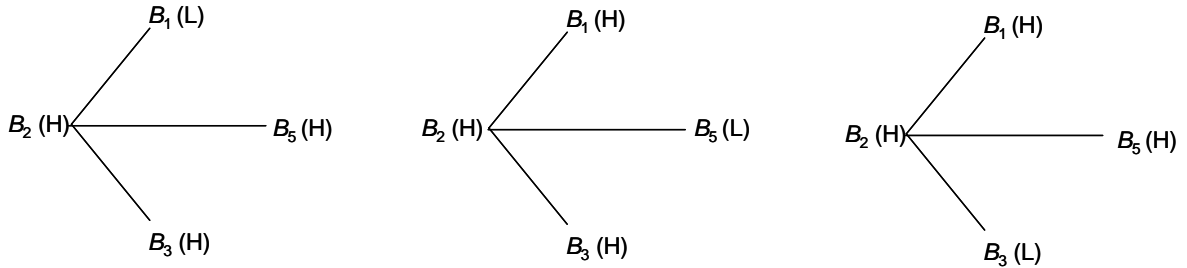
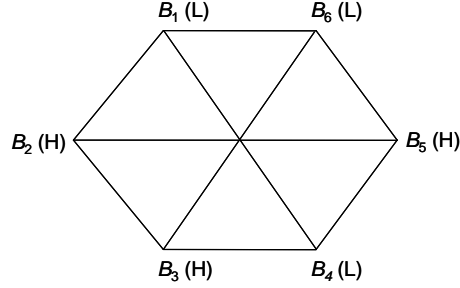


Figure 4.2: Uncertainty about which neighbours are of a different type

On the other hand, any link that connects two banks is private information for the respective institutions. Even though it is common knowledge that each bank i has n links, which nodes are at the end of these links is only known by i .⁷ This sort of incomplete information generates uncertainty about the minimum number of links that will connect banks of a different type. Banks are said to be of a different type if they will experience different liquidity shocks in period 1. In particular, a bank is of type H if it will face a high liquidity demand and a bank is of type L if it will face a low liquidity demand.

Example 2 Suppose that $n = 3$ and the network g is represented as bellow.

⁷This motivates our choice of 6 banks. In a 4 - bank setting, if n is common knowledge, each bank can make inferences and accurately guess the network structure.



For this structure, in period 1, there will be at most two banks each having exactly one neighbor that experiences a different fraction of early consumers, regardless of the states of the world realized. Hence, for any state of the world realized there will be at least 5 links that connect the H nodes and the L nodes. From the perspective of any bank i , however, it seems possible that each bank has exactly one neighbor of a different type, and thus the minimum number of links connecting nodes of a different type is 3.

In the case of a complete network, banks' environment simplifies considerably since most of the uncertainty is resolved. When the network is complete each bank will have with certainty 3 neighbors of a different type than itself. Moreover, every node is linked to every other node and thus there will be exactly 9 links connecting the H nodes and the L nodes, for any state of the world that is realized. The only uncertainty that banks have to consider concerns which of their neighbors will be of a different type.

4.2 Deposit allocations

Liquidity imbalances that occur in period 1 can be solved by the transfer of funds from banks of type L to banks of type H . When the probability of a bank failure is small, the allocation rule for deposits needs to meet two criteria. First, interbank deposits should be large enough to insure perfectly against any distribution of liquidity shocks realized at date 1. That is, after the transfer of funds takes place, each bank's cash holdings will exactly match the liquidity demand. And second, given that the liquidity shocks are hedged, the risk of a bank failure needs to be considered. Thus, the level of interbank deposits needs to be low enough to minimize the risk of contagion.

In order to meet the first criterion, the interbank system is considered to be at date 1 in the state when each bank has exactly $n - 2$ nodes of a different type. Note that uncertainty about the state of the world allows *one bank* to have exactly $n - 2$ neighbors

of a different type, while uncertainty about the network structure allows *all the banks*, to have each exactly $n - 2$ neighbors of different type. Thus, the allocation of deposits that ensure the transfer of liquidity from L nodes to H nodes, for any state of the world is that allocation that permits the transfer when each bank has exactly $n - 2$ neighbors of a different type. To satisfy the second criterion, banks need to divide z , the amount they will borrow (lend), among the $n - 2$ neighbors of a different type. Moreover, each bank may have any of their neighbors of a different type than itself.

To summarize, the allocation of deposits should minimize the *loss given default* associated to each link they have, for the *worst case scenario*.⁸ We consider the *worst case scenario* to be the state of the world for which each bank has exactly $n - 2$ nodes of a different type. Since for any pair ij , LGD_{ij} is decreasing in a_{ij} , the minimization problem yields an ex-ante optimal allocation of deposits exchanged at date 0 between any two banks of $\frac{z}{n-2}$.

Proposition 1 *Let g be a n -regular network of banks, with $n \geq 3$. The allocation rule for deposits that sets $a_{ij} = \frac{z}{n-2}$, for any pair of banks $ij \in g$, is feasible.*

Proof. The proof is provided in the appendix. ■

4.3 Optimality

In this section we examine whether the minimal feasible allocation rule for deposits is optimal for the risk of contagion. Moreover, we discuss when the ex-ante optimal allocation is also ex-post optimal. In other words, we are interested in how the network structure affects the trade-off between perfect insurance against liquidity shocks and contagion risk.

Given that banks choose an allocation rule for deposits that sets $a_{ij} = \frac{z}{n-2}$, the loss of any bank i given the default of any neighbor j of i is given by

$$LGD_{ij}^* = \left(\frac{z}{n-2} \right) \frac{C_1 - x - ry}{1 + (nz)/(n-2)} = z \frac{C_1 - x - ry}{n-2 + nz}$$

The following proposition relates the optimality of LGD^* to the degree of network completeness.

⁸These loss averse actions are entirely consistent with the usual behavior of banks. The use of VaR measure in practice is a sufficient evidence to support the assumption of loss aversion.

Proposition 2 *Let g be an incomplete n -regular network (i.e. $n = 3, 4$) and consider any realization of the liquidity shocks that allows at least one bank to have minimum $(n - 1)$ neighbors of a different type. Then there exists a feasible allocation of deposits a_{ij} such that $a_{ij} \frac{C_1 - x - ry}{1 + \sum_{k \in N_j} a_{jk}} < LGD_{ij}^*$, for any pair $ij \in g$.*

Proof. The proof is provided in the appendix. ■

Proposition 2 tells us that the allocation of deposits sub-optimal ex-post, for any realization of the state of the world that is *not* the worst case scenario. In other words, when the network is incomplete, the degree of interdependence between does not insure that the corresponding losses are minimal.

Corollary 2 *For $n = 3$, the allocations of deposits $a_{ij} = \frac{3z}{5}$, for any pair $ij \in g$, satisfies proposition 2. When $n = 4$, the allocations of deposits that satisfies proposition 2 is $a_{ij} = \frac{3z}{8}$.*

Proposition 2 discusses the case for $n = 3, 4$ and the next corollary treats the case of complete networks. We briefly explain what happens for $n = 1, 2$. A network degree larger than 3 insures that the network is connected. For $n < 3$, however, the network structure could be characterized by "islands"⁹. Moreover, the liquidity demand and the liquidity supply in the separate islands might be mismatched. This would create uncertainty about the aggregate fraction of early consumers as well. Anticipating this outcome, it may be optimal for banks not to exchange deposits in the first place.

Corollary 3 *Let \tilde{g} be the complete network. Then, there is no feasible allocation of deposits a_{ij} such that $a_{ij} \frac{C_1 - x - ry}{1 + \sum_{k \in N_j} a_{jk}} < LGD_{ij}^*$, for all pairs $ij \in \tilde{g}$.*

Proof. The proof is provided in the appendix. ■

To clarify, there is no allocation of deposits that reduces the loss of one bank without increasing the loss of another bank. The intuition behind corollary 3 relies on the fact that in a complete network the worst case scenario is realized for any distribution of the liquidity shocks.

This result is particularly important since it states that the complete network is the only network where the ex-ante optimal degree of interdependence is also ex-post optimal.

⁹For $n = 2$ the network could be structured in two 2-regular components. For $n = 1$ there is no connected network structure.

4.4 Varying asset portfolio

We have discussed above what implications the interbank linkages have for contagion risk, under the assumption that banks' portfolio is fixed. We have considered that the amount invested in the liquid asset, x , will be qC_1 , while the amount invested in the illiquid asset, y , will cover $(1 - q)C_2/R$. In other words, up to now, we have constrained banks to create linkages on the interbank market in order to insure against the liquidity shocks that will hit the economy in period 1. Moreover, by fixing the cash holdings of banks at date 1, we have imposed the dependency of each bank on the banking system to z .

Our assumption was reasonable. In fact, Allen and Gale (2000) show that the distribution (x, y) of the initial wealth in the liquid and illiquid asset is such that the expected utility of consumers is maximized. Any deviation from this distribution generates welfare losses for consumers. Nevertheless, it may be the case that, anticipating the failure of a bank and the consequent contagious losses, banks might decide on a different portfolio distribution. For instance, the higher the probability of a bank failure, the more banks prefer to hold cash reserves larger than qC_1 . A larger investment in the liquid asset reduces the amount banks need to borrow from the interbank market. Thus, banks might favor a lower degree of dependency, even though it means that they need to trade for this consumers' welfare.

Indeed, let the new portfolio distribution to be (\bar{x}, \bar{y}) , where $\bar{x} > x$ and $\bar{y} < y$, such that $\bar{x} + \bar{y} = 1$. This further implies that $\bar{x} = \bar{q}C_1$, with $\bar{q} \in (q, p_H]$, and the amount banks need to transfer on the interbank market will be $\bar{z}C_1 = (p_H - \bar{q})C_1$. Note that $\bar{z} > 0$ provided that $\bar{q} < p_H$. Hence, as long as banks hold a positive amount of interbank deposits, the degree of interdependence in an incomplete network will be sub-optimal.

5 Concluding Remarks

The problem of contagion within the banking system is a fairly debated issue. This paper studies whether the degree of interdependence that exists between banks supports systemic stability. In particular, we investigate how the network structure affects the trade-off between perfect insurance against liquidity shocks and contagion risk. Not only that in an incomplete network the losses caused by contagion are larger than in a complete network, as we knew from Allen and Gale (2000). In addition, an incomplete network generates

an environment of uncertainty such that the ex-ante optimal allocation of deposits is sub-optimal ex-post. It is, indeed, usually the case that in an incomplete information setting the ex-ante optimal outcomes are not also ex-post optimal. The point our paper raises is that it is exactly in an incomplete network where this setting of incomplete information is created. We show that in a complete network the uncertainty is resolved, and we conclude that a complete network favors an optimal degree of interdependence.

In the end we discuss the robustness of our results and draw a parallel to the empirical research on contagion. Our model extends naturally to more than 6 regions. Recall that what drives the results is that the allocation of deposits ensures perfectly the banking system against liquidity shocks, for any realization of the states of the world. More precisely, the allocation rule for deposits is such that losses due to contagion are minimal in the worst case scenario. When the network is incomplete, the allocation of deposits set this way turns out to be sub-optimal for any realization of the state of the world that is not the worst case scenario. In a complete network, however, the allocation of deposits always minimizes contagion risk since any state of the world implies the worst case scenario. This feature of complete networks versus incomplete network is independent of the actual number of nodes (regions).

The message this paper transmits is rather optimistic. When the network is complete, banks have the right incentives to choose the degree of interdependence for which the contagion risk is minimum. In short, in a complete network the contagion risk is very low. This result can be interpreted in the light of the empirical research on contagion, which consistently finds that the banking system demonstrates a high resilience to shocks. Recall that we use the same tool as the empirical papers to assess contagion risk. At the same time, the analyses performed in these papers are usually limited to a single country interbank market, where the network is likely to be complete. Our model can thus account as an explanation to support the empirical evidence.

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A Appendix

In order to prove Proposition 1 and 2, respectively, we need to introduce further notations.

Let Ω be the set of all possible state of the worlds¹⁰ and denote with ω an element of this set. Let \mathbf{H}^ω denote the set of banks of type H and \mathbf{L}^ω the set of banks of type L in the state of the world ω .

Let s_i^{cr} denote the number of neighbors of bank i that are of a different type than i and s_i^{in} denote the number of neighbors of bank i that are of the same type as i . For the remainder of the paper, we call s_i^{cr} the crossing degree of bank i and s_i^{in} the inner degree of bank i . If the network degree is n , than for every bank i we have $s_i^{cr} + s_i^{in} = n$. Moreover, since $s_i^{cr} \geq n - 2$, the following condition holds $n - 3 \leq s_i^{in} \leq 2$.

This notation is useful to understand that any state of the world can be expressed in terms of inner and crossing degree. We distinguish the following cases, independent of the network structure.

Case 1 $n = 3$.

For $n = 3$, any state of the world ω will be converted to one of the following 4 situations¹¹:

1. *For any bank $i \in \mathbf{H}^\omega$, $s_i^{in} = 2$.*
2. *There exists exactly one bank $i \in \mathbf{H}^\omega$ such that $s_i^{in} = 2$ and for any bank $j \in \mathbf{H}^\omega - \{i\}$ we have $s_j^{in} = 1$.*
3. *There exists exactly one bank $i \in \mathbf{H}^\omega$ such that $s_i^{in} = 0$ and for any bank $j \in \mathbf{H}^\omega - \{i\}$ we have $s_j^{in} = 1$.*
4. *For any bank $i \in \mathbf{H}^\omega$, $s_i^{in} = 0$.*

Any other possibility is excluded. For instance, consider a situation that allows two banks i and j to have a innner degree $s_i^{in} = s_j^{in} = 2$. Suppose that the link ij is created, than each bank needs one more link with a bank of the same type. This implies that the third bank k must have $s_k^{in} = 2$, which falls under situation 1.

Case 2 $n = 4$.

For $n = 4$, any state of the world ω will be converted to one of the following 2 situations:

¹⁰We established that $\mathbf{card}(\Omega) = \binom{6}{3}$, where $\mathbf{card}(\cdot)$ represents the cardinality of a set.

¹¹We discuss only the case of banks of type H . Due to symmetry, the case of banks of type L is analogous.

1. For any bank $i \in \mathbf{H}^\omega$, $s_i^{in} = 2$.

2. There exists exactly one bank $i \in \mathbf{H}^\omega$ such that $s_i^{in} = 2$ and for any bank $j \in \mathbf{H}^\omega - \{i\}$ we have $s_j^{in} = 1$.

A similar reasoning as above applies to exclude any other situation.

Case 3 $n = 5$.

When the network is complete, any state of the world ω will be converted to the following situation. For any bank $i \in \mathbf{H}^\omega$, $s_i^{in} = 2$.

It is easy to check that any other situation violates the regularity of the network.

Lemma 2 *Let ω be the realized state of the world. Then for any bank $i \in \mathbf{H}^\omega$ with a inner degree s_i^{in} and a crossing degree s_i^{cr} there exists a bank $k \in \mathbf{L}^\omega$ such that $s_k^{in} = s_i^{in}$ and $s_k^{cr} = s_i^{cr}$.*

Proof. The proof is based on the fact that $\sum_{i \in \mathbf{H}} s_i^{cr} = \sum_{j \in \mathbf{L}} s_j^{cr}$. Consequently, $\sum_{i \in \mathbf{H}} s_i^{in} = \sum_{j \in \mathbf{L}} s_j^{in}$. This implies that when the banks in \mathbf{H}^ω are in one of the situation described above, than it is necessary that the banks in \mathbf{L}^ω are in exactly the same situation. ■

We shall now continue with the proof of proposition 1 and 2, respectively.

Proposition 1 *Let g be a n -regular network of banks, with $n \geq 3$. The allocation rule for deposits that sets $a_{ij} = \frac{z}{n-2}$, for any pair of banks $ij \in g$, is feasible.*

Proof. In order to prove that a_{ij} is feasible we need to show that for any bank i and for any neighbor j of i there exist d_{ij} and d_{ji} such that $\left| \sum_{j \in N_i} d_{ji} - \sum_{j \in N_i} d_{ij} \right| C_1 = zC_1$ and $0 \leq d_{ij}, d_{ji} \leq a_{ij}$.

The proof is rather constructive. Let ω be the state of the world. Consider the network $\bar{g} = g - (\{ij\}_{i,j \in \mathbf{H}} \cup \{ij\}_{i,j \in \mathbf{L}})$, where ij represents the link between banks i and j . In other words, \bar{g} is the network formed from the initial network by deletion of links between banks of the same type. Thus, \bar{g} is the set of links that exist between banks of a different type. The total number of links in the network \bar{g} is $\sum_{i \in \mathbf{H}} s_i^{cr} = \sum_{j \in \mathbf{L}} s_j^{cr}$ which is larger than $3(n-2)$. In the network \bar{g} we further delete links such that each bank has exactly $(n-2)$ neighbors. Let \hat{g} be the new network where each bank has exactly $(n-2)$ links. The reader may check that there exists a network \hat{g} for any $n \geq 3$.

For any link $ij \in \hat{g}$, we set $d_{ij} = \frac{z}{n-2}$ if $i \in \mathbf{L}$ and $j \in \mathbf{H}$ and $d_{ij} = 0$ otherwise. Similarly, for any link $ij \in g$ and $ij \notin \hat{g}$, we set $d_{ij} = 0$. These transfers clearly satisfy $\left| \sum_{j \in N_i} d_{ji} - \sum_{j \in N_i} d_{ij} \right| C_1 = zC_1$, q.e.d. ■

Proposition 2 *Let g be an incomplete n -regular network (i.e. $n = 3, 4$) and consider any realization of the liquidity shocks that allows at least one bank to have minimum $(n-1)$ neighbors of a different type. Then there exists a feasible allocation of deposits a_{ij} such that $a_{ij} \frac{C_1 - x - ry}{1 + \sum_{k \in N_j} a_{jk}} < LGD_{ij}^*$, for any pair $ij \in g$.*

Proof. We treat the two cases $n = 3$ and $n = 4$ separately.

For $n = 3$, we consider the following allocation of deposits: $a_{ij} = \frac{3z}{5}$ for all pairs ij . Clearly, this allocation satisfies $a_{ij} \frac{C_1 - x - ry}{1 + \sum_{k \in N_j} a_{jk}} < LGD_{ij}^*$. We just need to show that $a_{ij} = \frac{3z}{5}$ is feasible for all the states of the world that allow at least one bank to have minimum $(n-1)$ neighbors of a different type. In order for at least one bank to have minimum 2 neighbors of a different type, the banking system needs to be in one of the situations 2 – 4 corresponding to case 1. Moreover, lemma 2 ensures that there are at least 2 banks, one of type H and one of type H , each having minimum 2 neighbors of a different type.

If the system is in situation 2, we construct the transfer of deposits in the following way. Let $k \in \mathbf{L}$ be the bank such that $s_k^{cr} = 1$, and let $l \in \mathbf{H}$ be the bank such that $s_l^{cr} = 1$. Consider the transfer of deposits $d_{ij} = \frac{3z}{5}$ if $i \in \mathbf{L}$ and $j \in \mathbf{H}$. Set $d_{ki} = \frac{z}{5}$ for any $i \in \mathbf{L} - \{k\}$ and $d_{jl} = \frac{z}{5}$ for any $j \in \mathbf{H} - \{l\}$. For all the other links set $d = 0$. These transfers satisfy $\left| \sum_{j \in N_i} d_{ji} - \sum_{j \in N_i} d_{ij} \right| C_1 = zC_1$ and $0 \leq d_{ij} \leq a_{ij}$ for any pair $ij \in g$.

If the system is in situation 3 and 4, in a similar manner as above, we construct the networks \hat{g}_3 and \hat{g}_4 , respectively. \hat{g}_3 is the network where for each bank i , $s_i^{cr} = 2$, while \hat{g}_4 is the network where for each bank i , $s_i^{cr} = 3$. In situation 3, for any link $ij \in \hat{g}_3$ we set the transfers to be $d_{ij} = \frac{z}{2}$ if $i \in \mathbf{L}$ and $j \in \mathbf{H}$ and $d_{ij} = 0$ otherwise. Similarly, for any link $ij \in g$ and $ij \notin \hat{g}_3$, we set $d_{ij} = 0$. These transfers satisfy $\left| \sum_{j \in N_i} d_{ji} - \sum_{j \in N_i} d_{ij} \right| C_1 = zC_1$ and $0 \leq d_{ij} \leq a_{ij}$ for any pair $ij \in g$. In situation 4, for any link $ij \in \hat{g}_4$ we set the transfers to be $d_{ij} = \frac{z}{3}$ if $i \in \mathbf{L}$ and $j \in \mathbf{H}$ and $d_{ij} = 0$ otherwise. Similarly, for any link $ij \in g$ and $ij \notin \hat{g}_4$, we set $d_{ij} = 0$. These transfers satisfy $\left| \sum_{j \in N_i} d_{ji} - \sum_{j \in N_i} d_{ij} \right| C_1 = zC_1$ and $0 \leq d_{ij} \leq a_{ij}$ for any pair $ij \in g$.

For $n = 4$, we consider the following allocation of deposits: $a_{ij} = \frac{3z}{8}$ for all pairs ij . Clearly, this allocation satisfies $a_{ij} \frac{C_1 - x - ry}{1 + \sum_{k \in N_j} a_{jk}} < LGD_{ij}^*$. We just need to show that

$a_{ij} = \frac{3z}{8}$ is feasible for all the states of the world that allow at least one bank to have minimum $(n - 1)$ neighbors of a different type. In order for at least one bank to have minimum 3 neighbors of a different type, the banking system needs to be in the situation 2 corresponding to case 2. When the system is in situation 2, we construct the transfer of deposits in the following way. Let $k \in \mathbf{L}$ be the bank such that $s_k^{cr} = 2$, and let $l \in \mathbf{H}$ be the bank such that $s_l^{cr} = 2$. Consider the transfer of deposits $d_{ij} = \frac{3z}{8}$ if $i \in \mathbf{L}$ and $j \in \mathbf{H}$. Set $d_{ki} = \frac{z}{8}$ for any $i \in \mathbf{L} - \{k\}$ and $d_{jl} = \frac{z}{8}$ for any $j \in \mathbf{H} - \{l\}$. For all the other links set $d = 0$. These transfers satisfy $\left| \sum_{j \in N_i} d_{ji} - \sum_{j \in N_i} d_{ij} \right| C_1 = zC_1$ and $0 \leq d_{ij} \leq a_{ij}$ for any pair $ij \in g$. q.e.d. ■

Corrolary3 Let \tilde{g} be the complete network. Than, there is no feasible allocation of deposits a_{ij} such that $a_{ij} \frac{C_1 - x - ry}{1 + \sum_{k \in N_j} a_{jk}} < LGD_{ij}^*$, for all pairs $ij \in \tilde{g}$.

Proof. We assume there is a feasible allocation of deposits a_{ij} such that $a_{ij} \frac{C_1 - x - ry}{1 + \sum_{k \in N_j} a_{jk}} < LGD_{ij}^*$, for all pairs $ij \in \tilde{g}$. Since \tilde{g} is the complete network $\sum_{k \in N_j} a_{jk} = \sum_{\substack{k=1 \\ k \neq j}}^6 a_{jk} = S_j$.

Since the inequality $a_{ij} \frac{C_1 - x - ry}{1 + \sum_{k \in N_j} a_{jk}} < LGD_{ij}^*$ holds, than we must have $\frac{a_{ij}}{1 + \sum_{k \in N_j} a_{jk}} < \frac{z}{3 + 5z}$ or $(3 + 5z)a_{ij} < z + zS_j$, for any $i, j \in \{1, 2, \dots, 6\}$. Keeping j fixed and aggregating these inequalities after i , we obtain: $(3 + 5z) \sum_{\substack{i=1 \\ i \neq j}}^6 a_{ij} < 5z + 5zS_j$. This yields further $S_j < \frac{5z}{3}$, $\forall j \in \{1, 2, \dots, 6\}$. In order for an allocation of deposits a_{ij} to be feasible a necessary condition is that it exists a pair $kl \in \tilde{g}$ such that $a_{kl} > \frac{z}{3}$. Since $S_k < \frac{5z}{3}$, we must have $\frac{a_{kl}}{1 + S_k} > \frac{z}{3} \frac{1}{1 + 5z/3}$ or $a_{kl} \frac{C_1 - x - ry}{1 + S_k} > LGD_{kl}^*$ which contradicts our initial assumption. q.e.d. ■

In the end we give a general proof for the connectedness property of n -regular networks that we employ in the proof of Lemma 1 .

Lemma 3 Let $M = \{1, 2, \dots, m\}$ be a set of nodes connected in a n -regular network g . If $n \geq m/2$, than the network g is connected and the maximum path length between any two nodes is 2.

Proof. Consider the node $i \in M$ and let $N(i) = \{i_1, i_2, \dots, i_n\}$ be the set of nodes directly connected with i . Than $\mathbf{card}(M - N(i)) \leq m/2$. Since any node $j \in M - N(i)$ has degree $n \geq m/2$ than for $\forall j \in M - N(i)$, $\exists i_l \in N(i)$ such that j and i_l are directly connected. This further implies that j and i are connected through a path of length 2. ■